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EUCLID'S ELEMENTS.

BOOKS I, II.

WITH NOTES, QUESTIONS,
GEOMETRICAL EXERCISES, &c.

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580. a. 252

EUCLID'S

ELEMENTS OF GEOMETRY,

THE FIRST TWO BOOKS,

CHIEFLY FROM THE TEXT OF DR. SIMSON,
WITH EXPLANATORY NOTES;
A SERIES OF QUESTIONS ON EACH BOOK;
AND A SELECTION OF GEOMETRICAL EXERCISES FROM THE
SENATE-HOUSE AND COLLEGE EXAMINATION
PAPERS; WITH HINTS, ETC.

BY

ROBERT POTTS, M.A.
TRINITY COLLEGE.

CORRECTED AND IMPROVED.

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PREFACE TO THE THIRD EDITION.

SOME time after the publication of an Octavo Edition of Euclid's Elements with Geometrical Exercises, &c., designed for the use of Academical Students; at the request of some schoolmasters of eminence, a duodecimo Edition of the Six Books was put forth on the same plan for the use of Schools. Soon after its appearance, Professor Christie, the Secretary of the Royal Society, in the Preface to his Treatise on Descriptive Geometry for the use of the Royal Military Academy, was pleased to notice these works in the following terms:— "When the greater Portion of this Part of the Course was printed, and had for some time been in use in the Academy, a new Edition of Euclid's Elements, by Mr. Robert Potts, M.A., of Trinity College, Cambridge, which is likely to supersede most others, to the extent, at least, of the Six Books, was published. From the manner of arranging the Demonstrations, this edition has the advantages of the symbolical form, and it is at the same time free from the manifold objections to which that form is open. The duodecimo edition of this Work, comprising only the first Six Books of Euclid, with Deductions from them, having been introduced at this Institution as a text-book, now renders any other Treatise on Plane Geometry unnecessary in our course of Mathematics."

For the very favourable reception which both Editions have met with, the Editor's grateful acknowledgements are due. It has been his desire in putting forth a revised Edition of the School Euclid, to render the work in some degree more worthy of the favour which the former editions have received. In the present Edition several errors and oversights have been corrected and some additions made to the notes: the questions on each book have been considerably augmented and a better arrangement of the Geometrical Exercises has been attempted: and lastly, some hints and remarks on them have been given to assist the learner. The additions made to the present Edition amount to more than fifty pages, and, it is hoped, that they will render the work more useful to the learner.

And here an occasion may be taken to quote the opinions of some able men respecting the use and importance of the Mathematical Sciences.

On the subject of Education in its most extensive sense, an ancient writer "directs the aspirant after excellence to commence with the Science of Moral Culture; to proceed next to Logic; next to Mathematics; next to Physics; and lastly, to Theology." Another writer *on Education* would place Mathematics before Logic, which (he *remarks)* "seems the preferable course: for by practising itself in t former, the mind becomes stored with distinctions; the faculties of constancy and firmness are established; and its rule is always to distinguish between cavilling and investigation—between *close reasoning* and *cross reasoning;* for the contrary of all which habits, those are for the most part noted, who apply themselves to Logic without studying in some department of Mathematics; taking noise and wrangling for proficiency, and thinking refutation accomplished by the instancing of a doubt. This will explain the inscription placed by Plato over the door of his house: 'Whoso knows not Geometry, let him not enter here.' On the precedence of Moral Culture, however, to all the other Sciences, the acknowledgement is general, and the agreement entire." The same writer recommends the study of the Mathematics, for the cure of "compound ignorance." "Of this," he proceeds to say, "the essence is opinion not agreeable to fact; and it necessarily involves another opinion, namely, that we are already possessed of knowledge. So that besides not knowing, we know not that we know not; and hence its designation of compound ignorance. In like manner, as of many chronic complaints and established maladies, no cure can be effected by physicians of the body: of this, no cure can be effected by physicians of the mind: for with a pre-supposal of knowledge in our own regard, the pursuit and acquirement of further knowledge is not to be looked for. The approximate cure, and one from which in the main much benefit may be anticipated, is to engage the patient in the study of measures (Geometry, computation, &c.); for in such pursuits the true and the false are separated by the clearest interval, and no room is left for the intrusions of fancy. From these the mind may discover the delight of certainty; and when, on returning to his own opinions, it finds in them no such sort of repose and gratification, it may discover their erroneous character, its ignorance may become simple, and a capacity for the acquirement of truth and virtue be obtained."

Lord Bacon, the founder of Inductive Philosophy, was not insensible of the high importance of the Mathematical Sciences, as appears in the following passage from his work on "The Advancement of Learning."

"The Mathematics are either pure or mixed. To the pure Mathematics are those sciences belonging which handle quantity determinate, merely severed from any axioms of natural philosophy; and these are *two,* Geometry, and Arithmetic; the one handling quantity continued, *and the other dissevered.* Mixed hath for subject some axioms or *parts of natural philosophy,* and considereth quantity determined, as it *is auxiliary and* incident unto them. For many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity, nor accommodated unto use with sufficient dexterity, without the aid and intervening of the Mathematics: of which sort are perspective, music, astronomy, cosmography, architecture, enginery, and divers others.

"In the Mathematics I can report no deficience, except it be that men do not sufficiently understand the excellent use of the pure Mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For, if the wit be dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect that it maketh a quick eye, and a body ready to put itself into all postures; so in the Mathematics, that use which is collateral and intervenient, is no less worthy than that which is principal and intended. And as for the mixed Mathematics, I may only make this prediction, that there cannot fail to be more kinds of them, as nature grows further disclosed."

How truly has this prediction been fulfilled in the subsequent advancement of the Mixed Sciences, and in the applications of the pure Mathematics to Natural Philosophy!

Dr. Whewell, in his "Thoughts on the Study of Mathematics," has maintained, that mathematical studies judiciously pursued, form one of the most effective means of developing and cultivating the reason: and that "the object of a *liberal education* is to develope the whole mental system of man;—to make his speculative inferences coincide with his practical convictions;—to enable him to render a reason for the belief that is in him, and not to leave him in the condition of Solomon's sluggard, who is wiser in his own conceit than seven men that *can* render a reason." And in his more recent work entitled, "Of a Liberal Education, &c." he has more fully shewn the importance of Geometry as one of the most effectual instruments of intellectual education. In page 55 he thus proceeds:—"But besides the value of Mathematical Studies in Education, as a perfect example and complete exercise of demonstrative reasoning; Mathematical Truths have this additional recommendation, that they have always been referred to, by each successive generation of thoughtful and cultivated men, as examples of truth and of demonstration; and have thus become standard points of reference, among cultivated men, whenever they speak of truth, knowledge, or proof. Thus Mathematics has not only a disciplinal but an historical interest. This is *peculiarly the* case with those portions of Mathematics which we have *mentioned.* We find geometrical proof adduced in illustration of th nature of reasoning, in the earliest speculations on this subject, the Dialogues of Plato; we find geometrical proof one of the main subjects of discussion in some of the most recent of such speculations, as those of Dugald Stewart and his contemporaries. The recollection of the truths of Elementary Geometry has, in all ages, given a meaning and a reality to the best attempts to explain man's power of arriving at truth. Other branches of Mathematics have, in like manner, become recognized examples, among educated men, of man's powers of attaining truth."

Dr. Pemberton, in the preface to his view of Sir Isaac Newton's Discoveries, makes mention of the circumstance, "that Newton used to speak with regret of his mistake, at the beginning of his Mathematical Studies, in having applied himself to the works of Descartes and other Algebraical writers, before he had considered the Elements of Euclid with the attention they deserve."

To these we may subjoin the opinion of Mr. John Stuart Mill, which he has recorded in his invaluable System of Logic, (Vol. II. p. 180) in the following terms. "The value of Mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, &c.) consists in the applicability not of its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general; and the applications of Mathematics to the simpler branches of physics, furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex. These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding, with Plato, one who is ἀγεωμέτρητος, as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy."

In addition to these authorities it may be remarked, that the new Regulations which were confirmed by a Grace of the Senate on the 11th of May, 1846, assign to Geometry and to Geometrical methods, a more important place in the Examinations both for Honors and for the Ordinary Degree in this University.

TRINITY COLLEGE,
March 1, 1850.

R. P.

The supplement to the School Euclid (about forty-eight pages) has been incorporated with this impression of the Fifth Edition.

TRINITY COLLEGE,
October, 1863.

EUCLID'S
ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

I.

A POINT is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

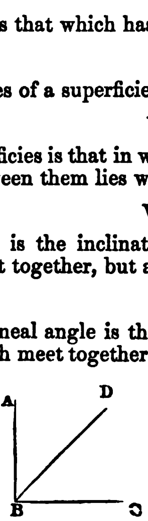
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

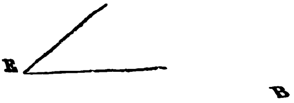
VIII.

A plane angle is the inclination of two lines to each other in a plane, which meet together, but are not in the same direction.

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

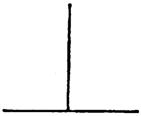




N.B. If there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle at *E*: but when several angles are at one point *B*, either of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of these straight lines, and the other upon the other line. Thus the angle which is contained by the straight lines *AB*, *CB*, is named the angle *ABC*, or *CBA*; that which is contained by *AB*, *DB*, is named the angle *ABD*, or *DBA*; and that which is contained by *DB*, *CB*, is called the angle *DBC*, or *CBD*.

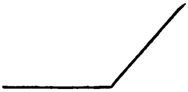
X.

When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



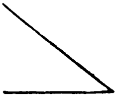
XI.

An obtuse angle is that which is greater than a right angle.



XII.

An acute angle is that which is less than a right angle.



XIII.

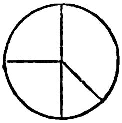
A term or boundary is the extremity of any thing.

XIV.

A figure is that which is enclosed by one or more boundaries.

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

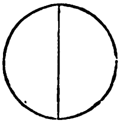


XVI.

And this point is called the center of the circle.

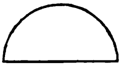
XVII.

A diameter of a circle is a straight line drawn through the center, and terminated both ways by the circumference.



XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.



XIX.

The center of a semicircle is the same with that of the circle.

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

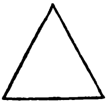
Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines.

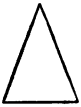
XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.



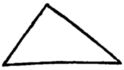
XXV.

An isosceles triangle is that which has two sides equal.



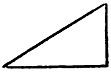
XXVI.

A scalene triangle is that which has three unequal sides.



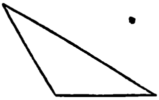
XXVII.

A right-angled triangle is that which has a right angle.



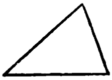
XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.



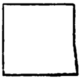
XXIX.

An acute-angled triangle is that which has three acute angles.



XXX.

Of quadrilateral or four-sided figures, a square has all its sides equal and all its angles right angles.

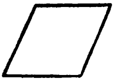


XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.

XXXII.

A rhombus has all its sides equal, but its angles are not right angles.



XXXIII.

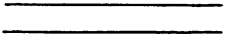
A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four-sided figures besides these, are called Trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways, do not meet.



A.

A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter, or the diagonal is the straight line joining two of its opposite angles.

POSTULATES.

I.

Let it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any center, at any distance *from that center.*

AXIOMS.

I.

THINGS which are equal to the same thing are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot enclose a space.

XI.

All right angles are equal to one another.

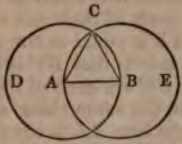
XII.

If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles; these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let *AB* be the given straight line.
It is required to describe an equilateral triangle upon *AB*.

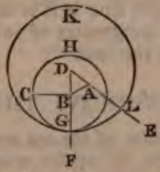


From the center *A*, at the distance *AB*, describe the circle *BCD*; (post. 3.)
from the center *B*, at the distance *BA*, describe the circle *ACE*;
and from *C*, one of the points in which the circles cut one another,
draw the straight lines *CA*, *CB* to the points *A*, *B*. (post. 1.)
Then *ABC* shall be an equilateral triangle.
Because the point *A* is the center of the circle *BCD*,
therefore *AC* is equal to *AB*; (def. 15.)
and because the point *B* is the center of the circle *ACE*,
therefore *BC* is equal to *AB*;
but it has been proved that *AC* is equal to *AB*;
therefore *AC*, *BC* are each of them equal to *AB*;
but things which are equal to the same thing are equal to one another;
therefore *AC* is equal to *BC*; (ax. 1.)
wherefore *AB*, *BC*, *CA* are equal to one another:
and the triangle *ABC* is therefore equilateral,
and it is described upon the given straight line *AB*.
Which was required to be done.

PROPOSITION II. PROBLEM.

From a given point, to draw a straight line equal to a given straight line.

Let *A* be the given point, and *BC* the given straight line.
It is required to draw from the point *A*, a straight line equal to *BC*.



From the point *A* to *B* draw the straight line *AB*; (post. 1.)
upon *AB* describe the equilateral triangle *ABD*, (I. 1.)
and produce the straight lines *DA*, *DB* to *E* and *F*; (post. 2.)
from the center *B*, at the distance *BC*, describe the circle *CGH*, (post. 3.) cutting *DF* in the point *G*:
and from the center *D*, at the distance DG, describe the circle GKL, *cutting AE* in the point L.

Then the straight line AL shall be equal to BC.
Because the point B is the center of the circle CGH,
therefore BC is equal to BG; (def. 15.)
and because D is the center of the circle GKL,
therefore DL is equal to DG,
and DA, DB parts of them are equal; (I. 1.)
therefore the remainder AL is equal to the remainder BG; (ax. 3.)
but it has been shewn that BC is equal to BG,
wherefore AL and BC are each of them equal to BG;
and things that are equal to the same thing are equal to one another;
therefore the straight line AL is equal to BC. (ax. 1.)

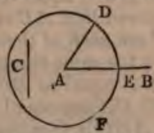
Wherefore from the given point A, a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB the greater, a part equal to C, the less.



From the point A draw the straight line AD equal to C; (I. 2.) and from the center A, at the distance AD, describe the circle DEF (post. 3.) cutting AB in the point E.

Then AE shall be equal to C.
Because A is the center of the circle DEF,
therefore AE is equal to AD; (def. 15.)
but the straight line C is equal to AD; (constr.)
whence AE and C are each of them equal to AD;
wherefore the straight line AE is equal to C. (ax. 1.)

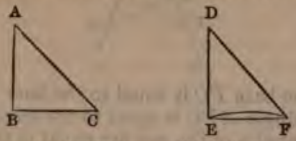
And therefore from AB the greater of two straight lines, a part AE has been cut off equal to C, the less. Which was to be done.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases or third sides *equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and the included angle BAC equal to the included angle EDF.

Then shall the base BC be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.



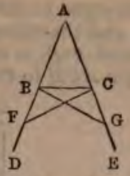
For, if the triangle ABC be applied to the triangle DEF,
so that the point A may be on D, and the straight line AB on DE;
then the point B shall coincide with the point E,
because AB is equal to DE;
and AB coinciding with DE,
the straight line AC shall fall on DF,
because the angle BAC is equal to the angle EDF;
therefore also the point C shall coincide with the point F,
because AC is equal to DF;
but the point B was shewn to coincide with the point E;
wherefore the base BC shall coincide with the base EF;
because the point B coinciding with E, and C with F,
if the base BC do not coincide with the base EF, the two straight lines BC and EF would enclose a space, which is impossible. (ax. 10.)
Therefore the base BC does coincide with EF, and is equal to it;
and the whole triangle ABC coincides with the whole triangle DEF, and is equal to it;
also the remaining angles of one triangle coincide with the remaining angles of the other, and are equal to them,
viz. the angle ABC to the angle DEF,
and the angle ACB to DFE.

Therefore, if two triangles have two sides of the one equal to two sides, &c. Which was to be demonstrated.

PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles triangle of which the side AB is equal to AC,
and let the equal sides AB, AC be produced to D and E.
Then the angle ABC shall be equal to the angle ACB,
and the angle DBC to the angle ECB.
In BD take any point F;
from AE the greater, cut off AG equal to AF the less, (I. 3.)
and join FC, GB.
Because AF is equal to AG, (constr.) and AB to AC; (hyp.)
the two sides FA, AC are equal to the two GA, AB, each to each;
and they contain the angle FAG common to the two triangles AFC, AGB;

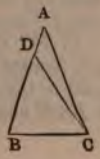


therefore the base *FC* is equal to the base *GB*, (I. 4.)
and the triangle *AFC* is equal to the triangle *AGB*,
also the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite;
viz. the angle *ACF* to the angle *ABG*,
and the angle *AFC* to the angle *AGB*.
And because the whole *AF* is equal to the whole *AG*,
of which the parts *AB*, *AC*, are equal;
therefore the remainder *BF* is equal to the remainder *CG*; (ax. 3.)
and *FC* has been proved to be equal to *GB*;
hence, because the two sides *BF*, *FC* are equal to the two *CG*, *GB*, each to each;
and the angle *BFC* has been proved to be equal to the angle *CGB*,
also the base *BC* is common to the two triangles *BFC*, *CGB*;
wherefore these triangles are equal, (I. 4.)
and their remaining angles, each to each, to which the equal sides are opposite;
therefore the angle *FBC* is equal to the angle *GCB*,
and the angle *BCF* to the angle *CBG*.
And, since it has been demonstrated,
that the whole angle *ABG* is equal to the whole *ACF*,
the parts of which, the angles *CBG*, *BCF* are also equal;
therefore the remaining angle *ABC* is equal to the remaining angle *ACB*,
which are the angles at the base of the triangle *ABC*;
and it has also been proved,
that the angle *FBC* is equal to the angle *GCB*,
which are the angles upon the other side of the base.
Therefore the angles at the base, &c. Q.E.D.
COR. Hence an equilateral triangle is also equiangular.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to each other; the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB*.
Then the side *AB* shall be equal to the side *AC*.



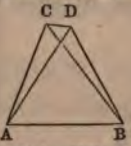
For, if AB be not equal to AC,
one of them is greater than the other.
If possible, let AB be greater than AC;
and from BA cut off BD equal to CA the less, (I. 3.) and join DC.
Then, in the triangles DBC, ABC,
because DB is equal to AC, and BC is common to both triangles,
the two sides DB, BC are equal to the two sides AC, CB, each to each;
and the angle DBC is equal to the angle ACB; (hyp.)
therefore the base DC is equal to the base AB, (I. 4.)
and the triangle DBC is equal to the triangle ABC,
the less equal to the greater, which is absurd. (ax. 9.)
Therefore AB is not unequal to AC, that is, AB is equal to AC.
Wherefore, if two angles, &c. Q.E.D.

COR. Hence an equiangular triangle is also equilateral.

PROPOSITION VII. THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, on the same base AB, and upon the same side of it, let there be two triangles ACB, ADB, which have their sides CA, DA, terminated in the extremity A of the base, equal to one another, and likewise their sides CB, DB, that are terminated in B.

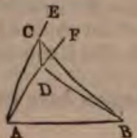


Join CD.

First. When the vertex of each of the triangles is without the other triangle.

Because AC is equal to AD in the triangle ACD,
therefore the angle ADC is equal to the angle ACD; (I. 5.)
but the angle ACD is greater than the angle BCD; (ax. 9.)
therefore also the angle ADC is greater than BCD;
much more therefore is the angle BDC greater than BCD.
Again, because the side BC is equal to BD in the triangle BCD, (hyp.)
therefore the angle BDC is equal to the angle BCD; (I. 5.)
but the angle BDC was proved greater than the angle BCD,
hence the angle BDC is both equal to, and greater than the angle BCD;
which is impossible.

Secondly. Let the vertex D of the triangle ADB fall within the triangle ACB.



Produce AC to E, and AD to F, and join CD.

Then because AC is equal to AD in the triangle ACD,

therefore the angles ECD, FDC upon the other side of the base CD, are equal to one another; (I. 5.)

but the angle ECD is greater than the angle BCD; (ax. 9.)

therefore also the angle FDC is greater than the angle BCD;

much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD in the triangle BCD,

therefore the angle BDC is equal to the angle BCD. (I. 5.)

but the angle BDC has been proved greater than BCD,

wherefore the angle BDC is both equal to, and greater than the angle BCD; which is impossible.

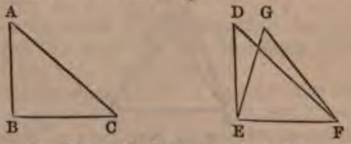
Thirdly. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base and on the same side of it, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and also the base BC equal to the base EF.



Then the angle BAC shall be equal to the angle EDF.

For, if the triangle ABC be applied to DEF,

so that the point B be on E, and the straight line BC on EF;

then because BC is equal to EF, (hyp.)

therefore the point C shall coincide with the point F;

wherefore BC coinciding with EF,

BA and AC shall coincide with ED, DF;

for, if the base BC coincide with the base EF, but the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation as EG, GF:

then, upon the same base, and upon the same side of it, there can be two triangles which have their sides which are terminated in one extremity of the base, equal to one another, and likewise those sides which are terminated in the other extremity; but this is impossible. (I. 7.)

Therefore, if the base BC coincide with the base EF,

the sides BA, AC cannot but coincide with the sides ED, DF;

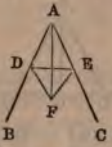
wherefore likewise the angle BAC coincides with the angle EDF, and *is equal to it.* (ax. 8.)

Therefore if two triangles have two sides, &c. Q.E.D.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle.
It is required to bisect it.

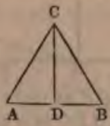


In AB take any point D;
from AC cut off AE equal to AD, (I. 3.) and join DE;
on the side of DE remote from A,
describe the equilateral triangle DEF (I. 1.), and join AF.
Then the straight line AF shall bisect the angle BAC.
Because AD is equal to AE, (constr.)
and AF is common to the two triangles DAF, EAF;
the two sides DA, AF, are equal to the two sides EA, AF, each to each;
and the base DF is equal to the base EF: (constr.)
therefore the angle DAF is equal to the angle EAF. (I. 8.)
Wherefore the angle BAC is bisected by the straight line AF. Q.E.F.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.
It is required to divide AB into two equal parts.
Upon AB describe the equilateral triangle ABC; (I. 1.)



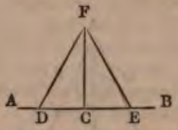
and bisect the angle ACB by the straight line CD meeting AB in the point D. (I. 9.)
Then AB shall be cut into two equal parts in the point D.
Because AC is equal to CB, (constr.)
and CD is common to the two triangles ACD, BCD;
the two sides AC, CD are equal to the two BC, CD, each to each;
and the angle ACD is equal to BCD; (constr.)
therefore the base AD is equal to the base BD. (I. 4.)
Wherefore the straight line AB is divided into two equal parts in the point D. Q.E.F.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let *AB* be the given straight line, and *C* a given point in it.

It is required to draw a straight line from the point *C* at right angles to *AB*.



In *AC* take any point *D*, and make *CE* equal to *CD*; (I. 3.)

upon *DE* describe the equilateral triangle *DEF* (I. 1), and join *CF*.

Then *CF* drawn from the point *C*, shall be at right angles to *AB*.

Because *DC* is equal to *EC*, and *FC* is common to the two triangles *DCF*, *ECF*;

the two sides *DC*, *CF* are equal to the two sides *EC*, *CF*, each to each;

and the base *DF* is equal to the base *EF*; (constr.)

therefore the angle *DCF* is equal to the angle *ECF*: (I. 8.)

and these two angles are adjacent angles.

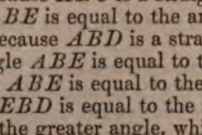
But when the two adjacent angles which one straight line makes with another straight line, are equal to one another, each of them is called a right angle: (def. 10.)

therefore each of the angles *DCF*, *ECF* is a right angle.

Wherefore from the given point *C*, in the given straight line *AB*, *FC* has been drawn at right angles to *AB*. Q.E.F.

COR. By help of this problem, it may be demonstrated that two straight lines cannot have a common segment.

If it be possible, let the segment *AB* be common to the two straight lines *ABC*, *ABD*.



From the point *B*, draw *BE* at right angles to *AB*; (I. 11.)

then because *ABC* is a straight line,

therefore the angle *ABE* is equal to the angle *EBC*. (def. 10.)

Similarly, because *ABD* is a straight line,

therefore the angle *ABE* is equal to the angle *EBD*;

but the angle *ABE* is equal to the angle *EBC*,

wherefore the angle *EBD* is equal to the angle *EBC*, (ax. 1.)

the less equal to the greater angle, which is impossible.

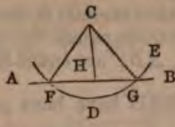
Therefore two straight lines cannot have a common segment.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let *AB* be the given straight line, which may be produced any length both ways, and let *C* be a point without it.

It is required to draw a straight line perpendicular to *AB* from the point *C*.



Upon the other side of *AB* take any point *D*,
and from the center *C*, at the distance *CD*, describe the circle *EGF*
meeting *AB*, produced if necessary, in *F* and *G*: (post. 3.)
bisect *FG* in *H* (I. 10.), and join *CH*.

Then the straight line *CH* drawn from the given point *C*, shall be perpendicular to the given straight line *AB*.

Join *FC*, and *CG*.
Because *FH* is equal to *HG*, (constr.)
and *HC* is common to the triangles *FHC*, *GHC*;
the two sides *FH*, *HC*, are equal to the two *GH*, *HC*, each to each;
and the base *CF* is equal to the base *CG*; (def. 15.)
therefore the angle *FHC* is equal to the angle *GHC*; (I. 8.)
and these are adjacent angles.

But when a straight line standing on another straight line, makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it. (def. 10.)

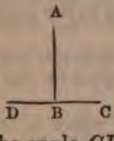
Therefore from the given point *C*, a perpendicular *CH* has been drawn to the given straight line *AB*. Q.E.F.

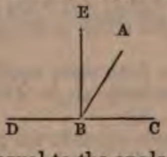
PROPOSITION XIII. THEOREM.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line *AB* make with *CD*, upon one side of it, the angles *CBA*, *ABD*.

Then these shall be either two right angles,
or, shall be together, equal to two right angles.





For if the angle *CBA* be equal to the angle *ABD*,
each of them is a right angle. (def. 10.)
But if the angle *CBA* be not equal to the angle *ABD*,
from the point *B* draw *BE* at right angles to CD. (I. 11.)
Then the angles *CBE*, EBD are two right angles. (def. 10.)

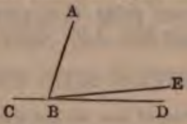
And because the angle *CBE* is equal to the angles *CBA*, *ABE*,
add the angle *EBD* to each of these equals;
therefore the angles *CBE*, *EBD* are equal to the three angles *CBA*, *ABE*, *EBD*. (ax. 2.)
Again, because the angle *DBA* is equal to the two angles *DBE*, *EBA*,
add to each of these equals the angle *ABC*;
therefore the angles *DBA*, *ABC* are equal to the three angles *DBE*, *EBA*, *ABC*.

But the angles *CBE*, *EBD* have been proved equal to the same three angles;
and things which are equal to the same thing are equal to one another;
therefore the angles *CBE*, *EBD* are equal to the angles *DBA*, *ABC*;
but the angles *CBE*, *EBD* are two right angles;
therefore the angles *DBA*, *ABC* are together equal to two right angles. (ax. 1.)
Wherefore, when a straight line, &c. Q.E.D.

PROPOSITION XIV. THEOREM.

If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles; then these two straight lines shall be in one and the same straight line.

At the point *B* in the straight line *AB*, let the two straight lines *BC*, *BD* upon the opposite sides of *AB*, make the adjacent angles *ABC*, *ABD* together equal to two right angles.
Then *BD* shall be in the same straight line with *BC*.



For, if *BD* be not in the same straight line with *BC*,
if possible, let *BE* be in the same straight line with it.
Then because *AB* meets the straight line *CBE*;
therefore the adjacent angles *CBA*, *ABE* are equal to two right angles; (I. 13.)
but the angles *CBA*, *ABD* are equal to two right angles; (hyp.)
therefore the angles *CBA*, *ABE* are equal to the angles *CBA*, *ABD*: (ax. 1.)
take away from these equals the common angle *CBA*,
therefore the remaining angle *ABE* is equal to the remaining angle *ABD*; (ax. 3.)
the less angle equal to the greater, which is impossible:
therefore *BE* is not in the same straight line with *BC*.

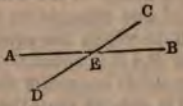
And in the same manner it may be demonstrated, that no other can be in the same straight line with it but *BD*, which therefore is in the same straight line with *BC*.
Wherefore, if at a point, &c. Q.E.D.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical, or opposite *angles shall be equal.*

Let the two straight lines *AB*, *CD* cut one another in the point *E*.

Then the angle *AEC* shall be equal to the angle *DEB*, and the angle *CEB* to the angle *AED*.



Because the straight line *AE* makes with *CD* at the point *E*, the adjacent angles *CEA*, *AED*;

these angles are together equal to two right angles. (I. 13.)

Again, because the straight line *DE* makes with *AB* at the point *E*, the adjacent angles *AED*, *DEB*;

these angles also are equal to two right angles;

but the angles *CEA*, *AED* have been shewn to be equal to two right angles;

wherefore the angles *CEA*, *AED* are equal to the angles *AED*, *DEB*;

take away from each the common angle *AED*,

and the remaining angle *CEA* is equal to the remaining angle *DEB*. (ax. 3.)

In the same manner it may be demonstrated, that the angle *CEB* is equal to the angle *AED*.

Therefore, if two straight lines cut one another, &c. Q.E.D.

Cor. 1. From this it is manifest, that, if two straight lines cut each other, the angles which they make at the point where they cut, are together equal to four right angles.

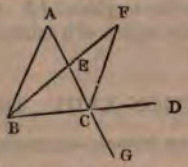
Cor. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let *ABC* be a triangle, and let the side *BC* be produced to *D*.

Then the exterior angle *ACD* shall be greater than either of the interior opposite angles *CBA* or *BAC*.



Bisect *AC* in *E*, (I. 10.) and join BE;

produce BE to F, making EF equal to BE, (I. 3.) and join FC.

Because AE is equal to EC, and BE to EF; (constr.)
the two sides AE, EB are equal to the two CE, EF, each to each, in the triangles ABE, CFE;
and the angle AEB is equal to the angle CEF,
because they are opposite vertical angles; (I. 15.)
therefore the base AB is equal to the base CF, (I. 4.)
and the triangle AEB to the triangle CEF,
and the remaining angles of one triangle to the remaining angles of the other, each to each, to which the equal sides are opposite;
wherefore the angle BAE is equal to the angle ECF;
but the angle ECD or ACD is greater than the angle ECF;
therefore the angle ACD is greater than the angle BAE or BAC.

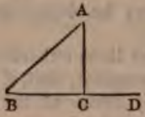
In the same manner, if the side BC be bisected, and AC be produced to G; it may be demonstrated that the angle BCG, that is, the angle ACD, (I. 15.) is greater than the angle ABC.

Therefore, if one side of a triangle, &c. Q.E.D.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.
Then any two of its angles together shall be less than two right angles.



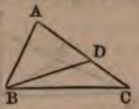
Produce any side BC to D.
Then because ACD is the exterior angle of the triangle ABC;
therefore the angle ACD is greater than the interior and opposite angle ABC; (I. 16.)
to each of these unequals add the angle ACB;
therefore the angles ACD, ACB are greater than the angles ABC, ACB;
but the angles ACD, ACB are equal to two right angles; (I. 13.)
therefore the angles ABC, ACB are less than two right angles.
In like manner it may be demonstrated,
that the angles BAC, ACB are less than two right angles,
as also the angles CAB, ABC.
Therefore any two angles of a triangle, &c. Q.E.D.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle is opposite to the greater angle.

Let ABC *be* a triangle, of which the side AC is greater than the *side* AB.

Then the angle ABC shall be greater than the angle ACB.



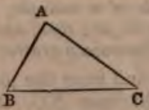
Since the side AC is greater than the side AB, (hyp.)
make AD equal to AB, (I. 3.) and join BD.
Then, because AD is equal to AB, in the triangle ABD,
therefore the angle ABD is equal to the angle ADB, (I. 5.)
but because the side CD of the triangle BDC is produced to A,
therefore the exterior angle ADB is greater than the interior and opposite angle DCB; (I. 16.)
but the angle ADB has been proved equal to the angle ABD,
therefore the angle ABD is greater than the angle DCB;
wherefore much more is the angle ABC greater than the angle ACB.
Therefore the greater side, &c. Q.E.D.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

Let ABC be a triangle of which the angle ABC is greater than the angle BCA.

Then the side AC shall be greater than the side AB.



For, if AC be not greater than AB,
AC must either be equal to, or less than AB;
if AC were equal to AB,
then the angle ABC would be equal to the angle ACB; (I. 5.)
but it is not equal; (hyp.)
therefore the side AC is not equal to AB.
Again, if AC were less than AB,
then the angle ABC would be less than the angle ACB; (I. 18.)
but it is not less, (hyp.)
therefore the side AC is not less than AB;
and AC has been shewn to be not equal to AB;
therefore AC is greater than AB.
Wherefore the greater angle, &c. Q.E.D.

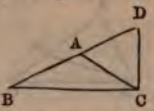
PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle.

Then any two sides of it together shall be greater than the third side, viz. the sides BA, AC greater than the side BC;

AB, *BC* greater than *AC*;
and *BC*, *CA* greater than *AB*.



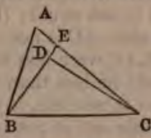
Produce the side *BA* to the point *D*,
make *AD* equal to *AC*, (I. 3.) and join *DC*.
Then because *AD* is equal to *AC*, (constr.)
therefore the angle *ACD* is equal to the angle *ADC*; (I. 5.)
but the angle *BCD* is greater than the angle *ACD*; (ax. 9.)
therefore also the angle *BCD* is greater than the angle *ADC*.
And because in the triangle *DBC*,
the angle *BCD* is greater than the angle *BDC*,
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side *DB* is greater than the side *BC*;
but *DB* is equal to *BA* and *AC*,
therefore the sides *BA* and *AC* are greater than *BC*.
In the same manner it may be demonstrated,
that the sides *AB*, *BC* are greater than *CA*;
also that *BC*, *CA* are greater than *AB*.
Therefore any two sides, &c. Q.E.D.

PROPOSITION XXI. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle; these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let *ABC* be a triangle, and from the points *B*, *C*, the ends of the side *BC*, let the two straight lines *BD*, *CD* be drawn to a point *D* within the triangle.

Then *BD* and *DC* shall be less than *BA* and *AC* the other two sides of the triangle,
but shall contain an angle *BDC* greater than the angle *BAC*.



Produce *BD* to meet the side *AC* in *E*.

Because two sides of a triangle are greater than the third side, (I. 20.) therefore the two sides *BA*, *AE* of the triangle *ABE* are greater than *BE*;

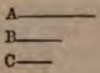
to each of these unequals add *EC*;
therefore the sides *BA*, *AC* are greater than *BE*, *EC*. (ax. 4.)
Again, because the two sides *CE*, *ED* of the triangle *CED* are *greater* than *DC*; (I. 20.)
add *DB* to each of these unequals;

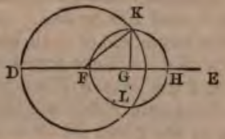
therefore the sides *CE*, *EB* are greater than *CD*, *DB*. (ax. 4.)
But it has been shewn that *BA*, *AC* are greater than *BE*, *EC*;
much more then are *BA*, *AC* greater than *BD*, *DC*.
Again, because the exterior angle of a triangle is greater than the interior and opposite angle; (I. 16.)
therefore the exterior angle *BDC* of the triangle *CDE* is greater than the interior and opposite angle *CED*;
for the same reason, the exterior angle *CED* of the triangle *ABE*
is greater than the interior and opposite angle *BAC*;
and it has been demonstrated,
that the angle *BDC* is greater than the angle *CEB*;
much more therefore is the angle *BDC* greater than the angle *BAC*.
Therefore, if from the ends of the side, &c. Q.E.D.

PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let *A*, *B*, *C* be the three given straight lines,
of which any two whatever are greater than the third, (I. 20.)
namely, *A* and *B* greater than *C*;
A and *C* greater than *B*;
and *B* and *C* greater than *A*.
It is required to make a triangle of which the sides shall be equal to *A*, *B*, *C*, each to each.





Take a straight line *DE* terminated at the point *D*, but unlimited towards *E*,
make *DF* equal to *A*, *FG* equal to *B*, and *GH* equal to *C*; (I. 3.)
from the center *F*, at the distance *FD*, describe the circle *DKL*; (post 3.)
from the center *G*, at the distance *GH*, describe the circle *HLK*;
from *K* where the circles cut each other, draw *KF*, *KG* to the points *F*, *G*;
Then the triangle *KFG* shall have its sides equal to the three straight lines *A*, *B*, *C*.
Because the point *F* is the center of the circle *DKL*,
therefore *FD* is equal to *FK*; (def. 15.)
but *FD* is equal to the straight line *A*;
therefore *FK* is equal to *A*.
Again, because *G* is the center of the circle *HKL*;
therefore *GH* is equal to *GK*, (def. 15.)
but *GH* is equal to *C*;
therefore also *GK* is equal to C; (ax. 1.)
and *FG* is equal to B;

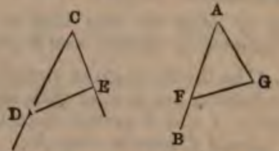
therefore the three straight lines *KF*, *FG*, *GK*, are respectively equal to the three, *A*, *B*, *C*:
and therefore the triangle *KFG* has its three sides *KF*, *FG*, *GK*, equal to the three given straight lines *A*, *B*, *C*. Q.E.F.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let *AB* be the given straight line, and *A* the given point in it, and *DCE* the given rectilineal angle.

It is required, at the given point *A* in the given straight line *AB*, to make an angle that shall be equal to the given rectilineal angle *DCE*.

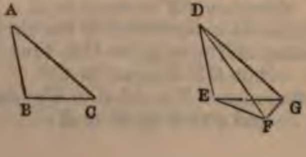


In *CD*, *CE*, take any points *D*, *E*, and join *DE*;
on *AB*, make the triangle *AFG*, the sides of which shall be equal to the three straight lines *CD*, *DE*, *EC*, so that *AF* be equal to *CD*, *AG* to *CE*, and *FG* to *DE*. (I. 22.)
Then the angle *FAG* shall be equal to the angle *DCE*.
Because *FA*, *AG* are equal to *DC*, *CE*, each to each,
and the base *FG* is equal to the base *DE*;
therefore the angle *FAG* is equal to the angle *DCE*. (I. 8.)
Wherefore, at the given point *A* in the given straight line *AB*, the angle *FAG* is made equal to the given rectilineal angle *DCE*. Q.E.F.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle, shall be greater than the base of the other.

Let *ABC*, *DEF* be two triangles, which have the two sides *AB*, *AC*, equal to the two *DE*, *DF*, each to each, namely, *AB* equal to *DE*, and *AC* to *DF*; but the angle *BAC* greater than the angle *EDF*.
Then the base *BC* shall be greater than the base *EF*.

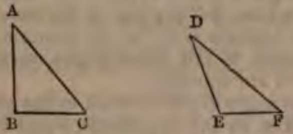


Of the two sides DE, DF, let DE be not greater than DF,
at the point D, in the line DE, and on the same side of it as DF,
make the angle EDG equal to the angle BAC; (I. 23.)
make DG equal to DF or AC, (I. 3.) and join EG, GF.
Then, because DE is equal to AB, and DG to AC,
the two sides DE, DG are equal to the two AB, AC, each to each,
and the angle EDG is equal to the angle BAC;
therefore the base EG is equal to the base BC. (I. 4.)
And because DG is equal to DF in the triangle DFG,
therefore the angle DFG is equal to the angle DGF; (I. 5.)
but the angle DGF is greater than the angle EGF; (ax. 9.)
therefore the angle DFG is also greater than the angle EGF;
much more therefore is the angle EFG greater than the angle EGF.
And because in the triangle EFG, the angle EFG is greater than the angle EGF,
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side EG is greater than the side EF;
but EG was proved equal to BC;
therefore BC is greater than EF.
Wherefore, if two triangles, &c. Q.E.D.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC, equal to the two sides DE, DF, each to each, namely, AB equal to DE, and AC to DF: but the base BC greater than the base EF.
Then the angle BAC shall be greater than the angle EDF.



For, if the angle BAC be not greater than the angle EDF,
it must either be equal to it, or less than it.
If the angle BAC were equal to the angle EDF,
then the base BC would be equal to the base EF; (I. 4.)
but it is not equal, (hyp.)
therefore the angle BAC is not equal to the angle EDF.
Again, if the angle BAC were less than the angle EDF,
then the base BC would be less than the base EF; (I. 24.)
but it is not less, (hyp.)
therefore the angle BAC is not less than the angle EDF;
and it has been shewn, that the angle BAC is not equal to the angle EDF;
therefore the angle BAC is greater than the angle EDF.
Wherefore, if two triangles, &c. Q.E.D.

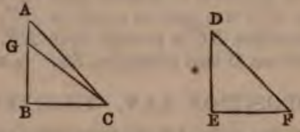
PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz. either the sides adjacent to the equal angles in each, or the sides opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA, equal to the angles DEF, EFD, each to each, namely, ABC to DEF, and BCA to EFD; also one side equal to one side.

First, let those sides be equal which are adjacent to the angles that are equal in the two triangles, namely, BC to EF.

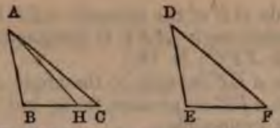
Then the other sides shall be equal, each to each, namely, AB to DE, and AC to DF, and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE,
one of them must be greater than the other.
If possible, let AB be greater than DE,
make BG equal to ED, (I. 3.) and join GC.
Then in the two triangles GBC, DEF,
because GB is equal to DE, and BC to EF, (hyp.)
the two sides GB, BC are equal to the two DE, EF, each to each;
and the angle GBC is equal to the angle DEF;
therefore the base GC is equal to the base DF, (I. 4.)
and the triangle GBC to the triangle DEF,
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle GCB is equal to the angle DFE;
but the angle ACB is, by the hypothesis, equal to the angle DFE;
wherefore also the angle GCB is equal to the angle ACB; (ax. 1.)
the less angle equal to the greater, which is impossible;
therefore AB is not unequal to DE,
that is, AB is equal to DE.
Hence, in the triangles ABC, DEF;
because AB is equal to DE, and BC to EF, (hyp.)
and the angle ABC is equal to the angle DEF; (hyp.)
therefore the base AC is equal to the base DF, (I. 4.)
and the third angle BAC to the third angle EDF.

Secondly, let the sides which are opposite to one of the equal angles in each triangle be equal to one another, namely, AB equal to DE.

Then in this case likewise the other sides shall be equal, AC to DF, *and* BC *to* EF; and also the third angle BAC to the third angle EDF.

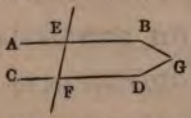


For if BC be not equal to EF,
one of them must be greater than the other.
If possible, let BC be greater than EF;
make BH equal to EF, (I. 3.) and join AH.
Then in the two triangles ABH, DEF,
because AB is equal to DE, and BH to EF,
and the angle ABH to the angle DEF; (hyp.)
therefore the base AH is equal to the base DF, (I. 4.)
and the triangle ABH to the triangle DEF,
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle BHA is equal to the angle EFD;
but the angle EFD is equal to the angle BCA; (hyp.)
therefore the angle BHA is equal to the angle BCA, (ax. 1.)
that is, the exterior angle BHA of the triangle AHC, is
equal to its interior and opposite angle BCA;
which is impossible; (I. 16.)
wherefore BC is not unequal to EF,
that is, BC is equal to EF.
Hence, in the triangles ABC, DEF;
because AB is equal to DE, and BC to EF, (hyp.)
and the included angle ABC is equal to the included angle DEF; (hyp.)
therefore the base AC is equal to the base DF, (I. 4.)
and the third angle BAC to the third angle EDF.
Wherefore, if two triangles, &c. Q.E.D.

PROPOSITION XXVII. THEOREM.

If a straight line falling on two other straight lines, make the alternate angles equal to each other; these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another.
Then AB shall be parallel to CD.



For, if AB be not parallel to CD,
then AB and CD being produced will meet, either towards A and C, or towards B and D.
Let AB, CD be produced and meet, if possible, towards B and D, in the point G,
then GEF is a triangle.

And because a side GE of the triangle GEF is produced to A,
therefore its exterior angle AEF is greater than the interior and opposite angle EFG; (I. 16.)
but the angle AEF is equal to the angle EFG; (hyp.)
therefore the angle AEF is greater than, and equal to, the angle EFG; which is impossible.

Therefore AB, CD being produced, do not meet towards B, D.

In like manner, it may be demonstrated, that they do not meet when produced towards A, C.

But those straight lines in the same plane, which meet neither way, though produced ever so far, are parallel to one another; (def. 35.)
therefore AB is parallel to CD.

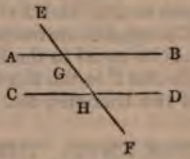
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXVIII. THEOREM.

If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD, upon the same side of the line EF; or make the two interior angles BGH, GHD on the same side together equal to two right angles.

Then AB shall be parallel to CD.



Because the angle EGB is equal to the angle GHD, (hyp.)
and the angle EGB is equal to the angle AGH, (I. 15.)
therefore the angle AGH is equal to the angle GHD; (ax. 1.)
and they are alternate angles,
therefore AB is parallel to CD. (I. 27.)

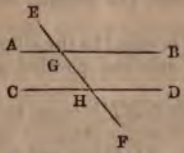
Again, because the angles BGH, GHD are together equal to two right angles, (hyp.)
and that the angles AGH, BGH are also together equal to two right angles; (I. 13.)
therefore the angles AGH, BGH are equal to the angles BGH, GHD; (ax. 1.)
take away from these equals, the common angle BGH;
therefore the remaining angle AGH is equal to the remaining angle GHD; (ax. 3.)
and they are alternate angles;
therefore AB is parallel to CD. (I. 27.)

Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line *EF* fall upon the parallel straight lines *AB*, *CD*.
Then the alternate angles *AGH*, *GHD* shall be equal to one another;
the exterior angle *EGB* shall be equal to the interior and opposite angle *GHD* upon the same side of the line *EF*;
and the two interior angles *BGH*, *GHD* upon the same side of *EF* shall be together equal to two right angles.



First. For, if the angle *AGH* be not equal to the alternate angle *GHD*, one of them must be greater than the other;
if possible, let *AGH* be greater than *GHD*,
then because the angle *AGH* is greater than the angle *GHD*,
add to each of these unequals the angle *BGH*;
therefore the angles *AGH*, *BGH* are greater than the angles *BGH*, *GHD*; (ax. 4.)
but the angles *AGH*, *BGH* are equal to two right angles; (I. 13.)
therefore the angles *BGH*, *GHD* are less than two right angles;
but those straight lines, which with another straight line falling upon them, make the two interior angles on the same side less than two right angles, will meet together if continually produced; (ax. 12.)
therefore the straight lines *AB*, *CD*, if produced far enough, will meet towards *B*, *D*;
but they never meet, since they are parallel by the hypothesis;
therefore the angle *AGH* is not unequal to the angle *GHD*,
that is, the angle *AGH* is equal to the alternate angle *GHD*.

Secondly. Because the angle *AGH* is equal to the angle *EGB*, (I. 15.)
and the angle *AGH* is equal to the angle *GHD*,
therefore the exterior angle *EGB* is equal to the interior and opposite angle *GHD*, on the same side of the line.

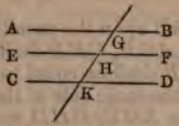
Thirdly. Because the angle *EGB* is equal to the angle *GHD*,
add to each of them the angle *BGH*;
therefore the angles *EGB*, *BGH* are equal to the angles *BGH*, *GHD*; (ax. 2.)
but *EGB*, *BGH* are equal to two right angles; (I. 13.)
therefore also the two interior angles *BGH*, *GHD* on the same side of the line are equal to two right angles. (ax. 1.)

Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to each other.

Let the straight lines AB, CD, be each of them parallel to EF.
Then shall AB be also parallel to CD.



Let the straight line GHK cut AB, EF, CD.

Then because GHK cuts the parallel straight lines AB, EF, in G, H:
therefore the angle AGH is equal to the alternate angle GHF. (I. 29.)

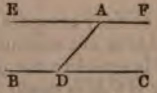
Again, because GHK cuts the parallel straight lines EF, CD, in H, K;
therefore the exterior angle GHF is equal to the interior angle HKD;
and it was shewn that the angle AGH is equal to the angle GHF;
therefore the angle AGH is equal to the angle GKD;
and these are alternate angles;
therefore AB is parallel to CD. (I. 27.)

Wherefore, straight lines which are parallel, &c. Q.E.D.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw, through the point A, a straight line parallel to the straight line BC.



In the line BC take any point D, and join AD;
at the point A in the straight line AD,
make the angle DAE equal to the angle ADC, (I. 23.) on the opposite side of AD;
and produce the straight line EA to F.

Then EF shall be parallel to BC.

Because the straight line AD meets the two straight lines EF, BC, and makes the alternate angles EAD, ADC, equal to one another,
therefore EF is parallel to BC. (I. 27.)

Wherefore, through the given point A, has been drawn a straight line EAF parallel to the given straight line BC. Q.E.F.

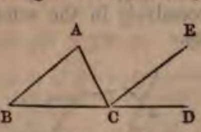
PROPOSITION XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D.

Then the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC:

and the three interior angles ABC, BCA, CAB shall be equal to two right angles.



Through the point C draw CE parallel to the side BA. (I. 31.)

Then because CE is parallel to BA, and AC meets them,

therefore the angle ACE is equal to the alternate angle BAC. (I. 29.)

Again, because CE is parallel to AB, and BD falls upon them, therefore the exterior angle ECD is equal to the interior and opposite angle ABC; (I. 29.)

but the angle ACE was shewn to be equal to the angle BAC;

therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC. (ax. 2.)

Again, because the angle ACD is equal to the two angles ABC, BAC,

to each of these equals add the angle ACB,

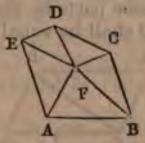
therefore the angles ACD and ACB are equal to the three angles ABC, BAC, and ACB. (ax. 2.)

but the angles ACD, ACB are equal to two right angles, (I. 13.)

therefore also the angles ABC, BAC, ACB are equal to two right angles. (ax. 1.)

Wherefore, if a side of any triangle be produced, &c. Q.E.D.

COR. 1. All the interior angles of any rectilineal figure together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure $ABCDE$ can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles.

Then, because the three interior angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides,

therefore all the angles of these triangles are equal to twice as many right angles as the figure has sides;

but the same angles of these triangles are equal to the interior angles of the figure together with the angles at the point F:

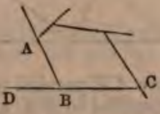
and the angles at the point *F*, which is the common vertex of all the triangles, are equal to four right angles, (I. 15. Cor. 2.)

therefore the same angles of these triangles are equal to the angles of the figure together with four right angles;

but it has been proved that the angles of the triangles are equal to twice as many right angles as the figure has sides;

therefore all the angles of the figure together with four right angles, are equal to twice as many right angles as the figure has sides.

COR. 2. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.



Since every interior angle *ABC* with its adjacent exterior angle *ABD*, is equal to two right angles, (I. 13.)

therefore all the interior angles, together with all the exterior angles, are equal to twice as many right angles as the figure has sides;

but it has been proved by the foregoing corollary, that all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides;

therefore all the interior angles together with all the exterior angles, are equal to all the interior angles and four right angles, (ax. 1.)

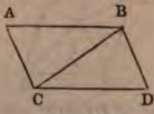
take from these equals all the interior angles,

therefore all the exterior angles of the figure are equal to four right angles. (ax. 3.)

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let *AB*, *CD* be equal and parallel straight lines,
and joined towards the same parts by the straight lines *AC*, *BD*.
Then *AC*, *BD* shall be equal and parallel.



Join *BC*.

Then because *AB* is parallel to *CD*, and *BC* meets them, therefore the angle *ABC* is equal to the alternate angle *BCD*; (I. 29.)

and because *AB* is equal to *CD*, and *BC* common to the two triangles *ABC*, *DCB*; the two sides *AB*, *BC*, are equal to the two *DC*, *CB*, each to each, and the angle *ABC* was proved to be equal to the angle *BCD*:

therefore the base *AC* is equal to the base *BD*, (I. 4.)
and the triangle *ABC* to the triangle BCD,

and the other angles to the other angles, each to each, to which the equal sides are opposite;

therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;

therefore AC is parallel to BD; (I. 27.)

and AC was shewn to be equal to BD.

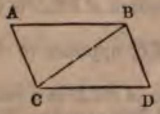
Therefore, straight lines which, &c. Q.E.D.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it, that is, divides it into two equal parts.

Let $ACDB$ be a parallelogram, of which BC is a diameter.

Then the opposite sides and angles of the figure shall be equal to one another; and the diameter BC shall bisect it.



Because AB is parallel to CD, and BC meets them,

therefore the angle ABC is equal to the alternate angle BCD. (I. 29.)

And because AC is parallel to BD, and BC meets them,

therefore the angle ACB is equal to the alternate angle CBD. (I. 29.)

Hence in the two triangles ABC, CBD,

because the two angles ABC, BCA in the one, are equal to the two angles BCD, CBD in the other, each to each;

and one side BC, which is adjacent to their equal angles, common to the two triangles;

therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, (I. 26.)

namely, the side AB to the side CD, and AC to BD, and the angle BAC to the angle BDC.

And because the angle ABC is equal to the angle BCD,

and the angle CBD to the angle ACB,

therefore the whole angle ABD is equal to the whole angle ACD; (ax. 2.)

and the angle BAC has been shewn to be equal to BDC;

therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter BC bisects it.

For since AB is equal to CD, and BC common, the two sides AB, BC, are equal to the two DC, CB, each to each,

and the angle ABC has been proved to be equal to the angle BCD;

therefore the triangle ABC is equal to the triangle BCD; (I. 4.) and the diameter BC divides the parallelogram ACDB into two equal parts.

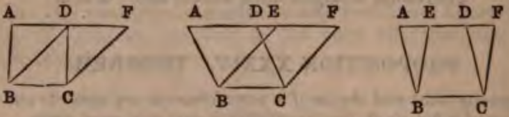
Q.E.D.

PROPOSITION XXXV. THEOREM.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms *ABCD*, *EBCF* be upon the same base *BC*, and between the same parallels *AF*, *BC*.

Then the parallelogram *ABCD* shall be equal to the parallelogram *EBCF*.



If the sides *AD*, *DF* of the parallelograms *ABCD*, *DBCF*, opposite to the base *BC*, be terminated in the same point *D*;
then it is plain that each of the parallelograms is double of the triangle *BDC*; (I. 34.)
and therefore the parallelogram *ABCD* is equal to the parallelogram *DBCF*. (ax. 6.)

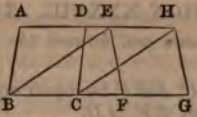
But if the sides *AD*, *EF*, opposite to the base *BC*, be not terminated in the same point;
Then, because *ABCD* is a parallelogram,
therefore *AD* is equal to *BC*; (I. 34.)
and for a similar reason, *EF* is equal to *BC*;
wherefore *AD* is equal to *EF*; (ax. 1.)
and *DE* is common;
therefore the whole, or the remainder *AE*, is equal to the whole, or the remainder *DF*; (ax. 2 or 3.)
and *AB* is equal to *DC*; (I. 24.)
hence in the triangles *EAB*, *FDC*,
because *FD* is equal to *EA*, and *DC* to *AB*,
and the exterior angle *FDC* is equal to the interior and opposite angle *EAB*; (I. 29.)
therefore the base *FC* is equal to the base *EB*, (I. 4.)
and the triangle *FDC* is equal to the triangle *EAB*.
From the trapezium *ABCF* take the triangle *FDC*,
and from the same trapezium take the triangle *EAB*,
and the remainders are equal, (ax. 3.)
therefore the parallelogram *ABCD* is equal to the parallelogram *EBCF*.
Therefore, parallelograms upon the same, &c. Q.E.D.

PROPOSITION XXXVI. THEOREM.

Parallelograms upon equal bases and between the same parallels, are equal to one another.

Let *ABCD*, *EFGH* be parallelograms upon equal bases *BC*, *FG*, and between the same parallels *AH*, *BG*.

Then the parallelogram *ABCD* shall be equal to the parallelogram *EFGH*.



Join BE, CH.

Then because BC is equal to FG, (hyp.) and FG to EH, (I. 34.)
therefore BC is equal to EH; (ax. 1.)

and these lines are parallels, and joined towards the same parts by the straight lines BE, CH;

but straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel; (I. 33.)

therefore BE, CH are both equal and parallel;
wherefore $EBCH$ is a parallelogram. (def. A.)

And because the parallelograms $ABCD$, $EBCH$, are upon the same base BC, and between the same parallels BC, AH;

therefore the parallelogram $ABCD$ is equal to the parallelogram $EBCH$. (I. 35.)

For the same reason, the parallelogram $EFGH$ is equal to the parallelogram $EBCH$;

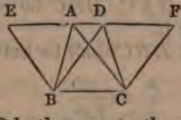
therefore the parallelogram $ABCD$ is equal to the parallelogram $EFGH$. (ax. 1.)

Therefore, parallelograms upon equal, &c. Q. E. D.

PROPOSITION XXXVII. THEOREM.

Triangles upon the same base and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC,
and between the same parallels AD, BC.
Then the triangle ABC shall be equal to the triangle DBC.



Produce AD both ways to the points E, F;
through B draw BE parallel to CA, (I. 31.)
and through C draw CF parallel to BD.

Then each of the figures $EBCA$, $DBCF$ is a parallelogram;
and $EBCA$ is equal to $DBCF$, (I. 35.) because they are upon the same base BC, and between the same parallels BC, EF.

And because the diameter AB bisects the parallelogram $EBCA$, therefore the triangle ABC is half of the parallelogram $EBCA$; (I. 34.)
also because the diameter DC bisects the parallelogram $DBCF$, therefore the triangle DBC is half of the parallelogram $DBCF$,
but the halves of equal things are equal; (ax. 7.)
therefore the triangle ABC is equal to the triangle DBC.

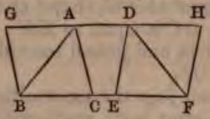
Wherefore, triangles, &c. Q.E.D.

PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases and between the same parallels, are equal to one another.

Let the triangles *ABC*, *DEF* be upon equal bases *BC*, *EF*, and between the same parallels *BF*, *AD*.

Then the triangle *ABC* shall be equal to the triangle *DEF*.



Produce *AD* both ways to the points *G*, *H*;
through *B* draw *BG* parallel to *CA*, (I. 31.)
and through *F* draw *FH* parallel to *ED*.
Then each of the figures *GBCA*, *DEFH* is a parallelogram;
and they are equal to one another, (I. 36.)
because they are upon equal bases *BC*, *EF*,
and between the same parallels *BF*, *GH*.

And because the diameter *AB* bisects the parallelogram *GBCA*, therefore the triangle *ABC* is the half of the parallelogram *GBCA*; (I. 34.)

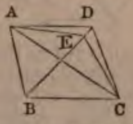
also, because the diameter *DF* bisects the parallelogram *DEFH*, therefore the triangle *DEF* is the half of the parallelogram *DEFH*;
but the halves of equal things are equal; (ax. 7.)
therefore the triangle *ABC* is equal to the triangle *DEF*.
Wherefore, triangles upon equal bases, &c. Q. E. D.

PROPOSITION XXXIX. THEOREM.

Equal triangles upon the same base and upon the same side of it, are between the same parallels.

Let the equal triangles *ABC*, *DBC* be upon the same base *BC*, and upon the same side of it.

Then the triangles *ABC*, *DBC* shall be between the same parallels.



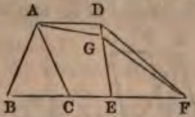
Join *AD*; then *AD* shall be parallel to *BC*.
For if *AD* be not parallel to *BC*,
if possible, through the point *A*, draw *AE* parallel to *BC*, (I. 31.) meeting *BD*, or *BD* produced, in *E*, and join *EC*.
Then the triangle *ABC* is equal to the triangle *EBC*, (I. 37.)
because they are upon the same base *BC*,
and between the same parallels *BC*, *AE*:
but the triangle *ABC* is equal to the triangle *DBC*; (hyp.)
therefore the triangle *DBC* is equal to the triangle *EBC*,

the greater triangle equal to the less, which is impossible:
therefore AE is not parallel to BC.
In the same manner it can be demonstrated,
that no other line drawn from A but AD is parallel to BC;
AD is therefore parallel to BC.
Wherefore, equal triangles upon, &c. Q.E.D.

PROPOSITION XL. THEOREM.

Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts.
Then they shall be between the same parallels.

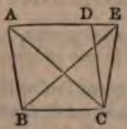


Join AD; then AD shall be parallel to BF.
For if AD be not parallel to BF,
if possible, through A draw AG parallel to BF, (I. 31.)
meeting ED, or ED produced in G, and join GF.
Then the triangle ABC is equal to the triangle GEF, (I. 38.)
because they are upon equal bases BC, EF,
and between the same parallels BF, AG;
but the triangle ABC is equal to the triangle DEF; (hyp.)
therefore the triangle DEF is equal to the triangle GEF, (ax. 1.)
the greater triangle equal to the less, which is impossible:
therefore AG is not parallel to BF.
And in the same manner it can be demonstrated,
that there is no other line drawn from A parallel to it but AD;
AD is therefore parallel to BF.
Wherefore, equal triangles upon, &c. Q.E.D.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE.
Then the parallelogram $ABCD$ shall be double of the triangle EBC.



Join AC.
Then the triangle ABC is equal to the triangle EBC, (I. 37.)

because they are upon the same base BC, and between the same parallels BC, AE.

But the parallelogram $ABCD$ is double of the triangle ABC, because the diameter AC bisects it; (I. 34.)

wherefore $ABCD$ is also double of the triangle EBC.

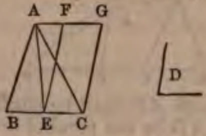
Therefore, if a parallelogram and a triangle, &c. Q.E.D.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Bisect BC in E, (I. 10.) and join AE;

at the point E in the straight line EC,

make the angle CEF equal to the angle D; (I. 23.)

through C draw CG parallel to EF, and through A draw AFG parallel to BC, (I. 31.) meeting EF in F, and CG in G.

Then the figure $CEFG$ is a parallelogram. (def. A.)

And because the triangles ABE, AEC are on the equal bases BE, EC, and between the same parallels BC, AG;

they are therefore equal to one another; (I. 38.)

and the triangle ABC is double of the triangle AEC;

but the parallelogram $FECG$ is double of the triangle AEC, (I. 41.) because they are upon the same base EC, and between the same parallels EC, AG;

therefore the parallelogram $FECG$ is equal to the triangle ABC, (ax. 6.) and it has one of its angles CEF equal to the given angle D.

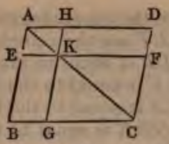
Wherefore, a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. Q.E.F.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC: and EH, GF the parallelograms about AC, *that is, through which AC passes:* also BK, KD the other parallelograms which make up the whole *figure $ABCD$,* which are therefore called the complements.

Then the complement BK shall be equal to the complement KD.



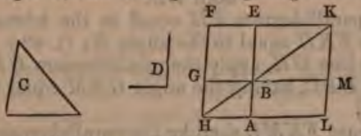
Because $ABCD$ is a parallelogram, and AC its diameter,
therefore the triangle ABC is equal to the triangle ADC. (I. 34.)
Again, because $EKHA$ is a parallelogram, and AK its diameter,
therefore the triangle AEK is equal to the triangle AHK; (I. 34.)
and for the same reason, the triangle KGC is equal to the triangle KFC.
Wherefore the two triangles AEK, KGC are equal to the two triangles AHK, KFC, (ax. 2.)
but the whole triangle ABC is equal to the whole triangle ADC;
therefore the remaining complement BK is equal to the remaining complement KD. (ax. 3.)
Wherefore the complements, &c. Q.E.D.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle.

It is required to apply to the straight line AB, a parallelogram equal to the triangle C, and having an angle equal to the angle D.



Make the parallelogram $BEFG$ equal to the triangle C,
and having the angle EBG equal to the angle D, (I. 42.)
so that BE be in the same straight line with AB;
produce FG to H,
through A draw AH parallel to BG or EF, (I. 31.) and join HB.
Then because the straight line HF falls upon the parallels AH, EF,
therefore the angles AHF, HFE are together equal to two right angles; (I. 29.)
wherefore the angles BHF, HFE are less than two right angles:
but straight lines which with another straight line, make the two interior angles upon the same side less than two right angles, do meet if produced far enough: (ax. 12.)
therefore HB, FE shall meet if produced;
let them be produced and meet in K,
through K draw KL parallel to EA or FH,
and produce HA, GB to meet KL in the points L, M.
Then $HLKF$ is a parallelogram, of which the diameter is HK;

and *AG*, *ME*, are the parallelograms about *HK*;
also *LB*, *BF* are the complements;
therefore the complement *LB* is equal to the complement *BF*; (I. 43.)
but the complement *BF* is equal to the triangle *C*; (constr.)
wherefore *LB* is equal to the triangle *C*.

And because the angle *GBE* is equal to the angle *ABM*, (I. 15.)
and likewise to the angle *D*; (constr.)
therefore the angle *ABM* is equal to the angle *D*. (ax. 1.)

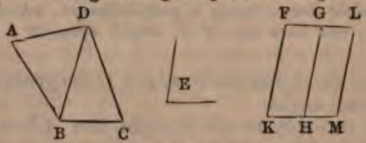
Therefore to the given straight line *AB*, the parallelogram *LB* has been applied, equal to the triangle *C*, and having the angle *ABM* equal to the given angle *D*. Q.E.F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let *ABCD* be the given rectilineal figure, and *E* the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the figure *ABCD*, and having an angle equal to the given angle *E*.



Join *DB*.

Describe the parallelogram *FH* equal to the triangle *ADB*, and having the angle *FKH* equal to the angle *E*; (I. 42.)
to the straight line *GH*, apply the parallelogram *GM* equal to the triangle *DBC*, having the angle *GHM* equal to the angle *E*. (I. 44.)

Then the figure *FKML* shall be the parallelogram required.

Because each of the angles *FKH*, *GHM*, is equal to the angle *E*,
therefore the angle *FKH* is equal to the angle *GHM*;
add to each of these equals the angle *KHG*;
therefore the angles *FKH*, *KHG* are equal to the angles *KHG*, *GHM*;
but *FKH*, *KHG* are equal to two right angles; (I. 29.)
therefore also *KHG*, *GHM* are equal to two right angles;
and because at the point *H*, in the straight line *GH*, the two straight lines *KH*, *HM*, upon the opposite sides of it, make the adjacent angles *KHG*, *GHM* equal to two right angles,
therefore *HK* is in the same straight line with *HM*. (I. 14.)
And because the line *HG* meets the parallels *KM*, *FG*,
therefore the angle *MHG* is equal to the alternate angle *HGF*; (I. 29.)
add to each of these equals the angle *HGL*;
therefore the angles *MHG*, *HGL* are equal to the angles *HGF*, *HGL*;
but the angles MHG, *HGL* are equal to two right angles; (I. 29.)
therefore also the angles HGF, *HGL* are equal to two right angles,
and therefore FG is in the same straight line with GL. (I. 14.)

And because KF is parallel to HG, and HG to ML,
therefore KF is parallel to ML; (I. 30.)
and FL has been proved parallel to KM,
wherefore the figure $FKML$ is a parallelogram;
and since the parallelogram HF is equal to the triangle ABD,
and the parallelogram GM to the triangle BDC;
therefore the whole parallelogram $KFLM$ is equal to the whole rectilineal figure $ABCD$.

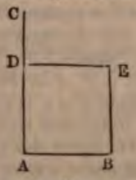
Therefore the parallelogram $KFLM$ has been described equal to the given rectilineal figure $ABCD$, having the angle FKM equal to the given angle E. Q.E.F.

COR. From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz. by applying to the given straight line a parallelogram equal to the first triangle ABD, (I. 44.) and having an angle equal to the given angle.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Let AB be the given straight line.



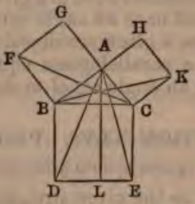
It is required to describe a square upon AB.
From the point A draw AC at right angles to AB; (I. 11.)
make AD equal to AB; (I. 3.)
through the point D draw DE parallel to AB; (I. 31.)
and through B, draw BE parallel to AD, meeting DE in E;
therefore $ABED$ is a parallelogram;
whence AB is equal to DE, and AD to BE; (I. 34.)
but AD is equal to AB,
therefore the four lines AB, BE, ED, DA are equal to one another,
and the parallelogram $ABED$ is equilateral.
It has likewise all its angles right angles;
since AD meets the parallels AB, DE,
therefore the angles BAD, ADE are equal to two right angles; (I. 29.)
but BAD is a right angle; (constr.)
therefore also ADE is a right angle.
But the opposite angles of parallelograms are equal; (I. 34.)
therefore each of the opposite angles ABE, BED is a right angle;
wherefore the figure $ABED$ is rectangular,
and it has been proved to be equilateral;
therefore the figure ABED is a square, (def. 30.)
and it is described upon the given straight line AB. Q.E.F.

COR. Hence, every parallelogram that has one of its angles a right angle, has all its angles right angles.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let *ABC* be a right-angled triangle, having the right angle *BAC*.
Then the square described upon the side *BC*, shall be equal to the squares described upon *BA*, *AC*.



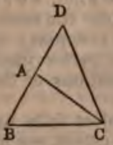
On *BC* describe the square *BDEC*, (I. 46.)
and on *BA*, *AC* the squares *GB*, *HC*;
through *A* draw *AL* parallel to *BD* or *CE*; (I. 31.)
and join *AD*, *FC*.
Then because the angle *BAC* is a right angle, (hyp.)
and that the angle *BAG* is a right angle, (def. 30.)
the two straight lines *AC*, *AG* upon the opposite sides of *AB*, make with it at the point *A*, the adjacent angles equal to two right angles;
therefore *CA* is in the same straight line with *AG*. (I. 14.)
For the same reason, *BA* and *AH* are in the same straight line.
And because the angle *DBC* is equal to the angle *FBA*,
each of them being a right angle,
add to each of these equals the angle *ABC*,
therefore the whole angle *ABD* is equal to the whole angle *FBC*. (ax. 2.)
And because the two sides *AB*, *BD*, are equal to the two sides *FB*, *BC*, each to each, and the included angle *ABD* is equal to the included angle *FBC*,
therefore the base *AD* is equal to the base *FC*, (I. 4.)
and the triangle *ABD* to the triangle *FBC*.
Now the parallelogram *BL* is double of the triangle *ABD*, (I. 41.)
because they are upon the same base *BD*, and between the same parallels *BD*, *AL*;
also the square *GB* is double of the triangle *FBC*,
because these also are upon the same base *FB*, and between the same parallels *FB*, *GC*.
But the doubles of equals are equal to one another; (ax. 6.)
therefore the parallelogram *BL* is equal to the square *GB*.
Similarly, by joining *AE*, *BK*, it can be proved,
that the parallelogram *CL* is equal to the square HC.

Therefore the whole square *BDEC* is equal to the two squares *GB*, *HC*; (ax. 2.)
and the square *BDEC* is described upon the straight line *BC*,
and the squares *GB*, *HC*, upon *AB*, *AC*:
therefore the square upon the side *BC*, is equal to the squares upon the sides *AB*, *AC*.
Therefore, in any right-angled triangle, &c. Q. E. D.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

Let the square described upon *BC*, one of the sides of the triangle *ABC*, be equal to the squares upon the other two sides, *AB*, *AC*.
Then the angle *BAC* shall be a right angle.



From the point *A* draw *AD* at right angles to *AC*, (I. 11.)
make *AD* equal to *AB*, and join *DC*.
Then, because *AD* is equal to *AB*,
the square on *AD* is equal to the square on *AB*;
to each of these equals add the square on *AC*;
therefore the squares on *AD*, *AC* are equal to the squares on *AB*, *AC*:
but the squares on *AD*, *AC* are equal to the square on *DC*, (I. 47.)
because the angle *DAC* is a right angle;
and the square on *BC*, by hypothesis, is equal to the squares on *BA*, *AC*;
therefore the square on *DC* is equal to the square on *BC*;
and therefore the side *DC* is equal to the side *BC*.
And because the side *AD* is equal to the side *AB*,
and *AC* is common to the two triangles *DAC*, *BAC*;
the two sides *DA*, *AC*, are equal to the two *BA*, *AC*, each to each;
and the base *DC* has been proved to be equal to the base *BC*;
therefore the angle *DAC* is equal to the angle *BAC*; (I. 8.)
but *DAC* is a right angle;
therefore also *BAC* is a right angle.
Therefore, if the square described upon, &c. Q. E. D.

NOTES TO BOOK I.

ON THE DEFINITIONS.

GEOMETRY is one of the most perfect of the deductive Sciences, and seems to rest on the simplest inductions from experience and observation.

The first principles of Geometry are therefore in this view consistent hypotheses founded on facts cognizable by the senses, and it is a subject of primary importance to draw a distinction between the conception of things and the things themselves. These hypotheses do not involve any property contrary to the real nature of the things, and consequently cannot be regarded as arbitrary, but in certain respects, agree with the conceptions which the things themselves suggest to the mind through the medium of the senses. The essential definitions of Geometry therefore being inductions from observation and experience, rest ultimately on the evidence of the senses.

It is by experience we become acquainted with the existence of individual forms of magnitudes; but by the mental process of abstraction, which begins with a particular instance, and proceeds to the general idea of all objects of the same kind, we attain to the general conception of those forms which come under the same general idea.

The essential definitions of Geometry express generalized conceptions of real existences in their most perfect ideal forms: the laws and appearances of nature, and the operations of the human intellect being supposed uniform and consistent.

But in cases where the subject falls under the class of simple ideas, the terms of the definitions so called, are no more than merely equivalent expressions. The simple idea described by a proper term or terms, does not in fact admit of definition properly so called. The definitions in Euclid's Elements may be divided into two classes, those which merely explain the meaning of the terms employed, and those, which, besides explaining the meaning of the terms, suppose the existence of the things described in the definitions.

Definitions in Geometry cannot be of such a form as to explain the nature and properties of the figures defined: it is sufficient that they give marks whereby the thing defined may be distinguished from every other of the same kind. It will at once be obvious, that the definitions of Geometry, one of the pure sciences, being abstractions of space, are not like the definitions in any one of the physical sciences. The discovery of any new physical facts may render necessary some alteration or modification in the definitions of the latter.

Def. I. Simson has adopted Theon's definition of a point. Euclid's definition is, *σημεῖον ἐστιν οὗ μέρος οὐδέν*, "A point is that, of which there is no part," or which cannot be parted or divided, as it is explained by Proclus. The Greek term *σημεῖον*, literally means, a visible *sign* or *mark* on a surface, in other words, *a physical point*. The English term *point*, means the sharp end of any thing, or a mark made by it. The word *point* comes from the Latin *punctum*, through the French word *point*. Neither of these terms, in its literal sense, appears to give a very exact notion of what is to be understood by a point in Geometry. Euclid's definition of a point merely expresses a negative property, which excludes the proper and literal meaning of the Greek term, as applied to denote a physical point, or a mark which is visible to the senses.

Pythagoras defined a point to be *μονὰς θέσιν ἔχουσα*, "a monad having position." By uniting the positive idea of position, with the negative *idea of defect of magnitude*, the conception of a point in Geometry may be rendered perhaps more intelligible. A point is defined to be that which has no magnitude, but position only.

Def. II. Every visible line has both length and breadth, and it is impossible to draw any line whatever which shall have no breadth. The definition requires the conception of the length only of the line to be considered, abstracted from, and independently of, all idea of its breadth.

Def. III. This definition renders more intelligible the exact meaning of the definition of a point: and we may add, that, in the Elements, Euclid supposes that the intersection of two lines is a point, and that two lines can intersect each other in one point only.

Def. IV. The straight line or right line is a term so clear and intelligible as to be incapable of becoming more so by formal definition. Euclid's definition is Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται, wherein he states it to lie *evenly*, or *equally*, or *upon an equality* (ἐξ ἴσου) between its extremities, and which Proclus explains as being stretched between its extremities, ἡ ἐπ' ἄκρων τεταμένη.

If the line be conceived to be drawn on a plane surface, the words ἐξ ἴσου may mean, that no part of the line which is called a straight line deviates either from one side or the other of the direction which is fixed by the extremities of the line; and thus it may be distinguished from a curved line, which does not lie, in this sense, evenly between its extreme points. If the line be conceived to be drawn in space, the words ἐξ ἴσου, must be understood to apply to every direction on every side of the line between its extremities.

Every straight line situated in a plane, is considered to have two sides; and when the direction of a line is known, the line is said to be given in position; also, when the length is known or can be found, it is said to be given in magnitude.

From the definition of a straight line, it follows, that two points fix a straight line in position, which is the foundation of the first and second postulates. Hence straight lines which are proved to coincide in two or more points, are called, "one and the same straight line," Prop. 14, Book I, or, which is the same thing, that "two straight lines cannot have a common segment," as Simson shews in his Corollary to Prop. 11, Book I.

The following definition of straight lines has also been proposed. "Straight lines are those which, if they coincide in any two points, coincide as far as they are produced." But this is rather a criterion of straight lines, and analogous to the eleventh axiom, which states that, "all right angles are equal to one another," and suggests that all straight lines may be made to coincide wholly, if the lines be equal; or partially, if the lines be of unequal lengths. A definition should properly be restricted to the description of the thing defined, as it exists, independently of any comparison of its properties or of tacitly assuming the existence of axioms.

Def. VII. Euclid's definition of a plane surface is Ἐπίπεδος ἐπιφάνειά ἐστιν ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται, "A plane surface is that which lies evenly or equally with the straight lines in it;" instead of which Simson has given the definition which was originally proposed by Hero the Elder. A plane superficies may be supposed to be situated in any position, and to be continued in every direction to any extent.

Def. VIII. Simson remarks that this definition seems to include the angles formed by two curved lines, or a curve and a straight line, as well as that formed by two straight lines.

Angles made by straight lines only, are treated of in Elementary Geometry.

Def. IX. It is of the highest importance to attain a clear conception of an angle, and of the sum and difference of two angles. The literal meaning of the term *angulus* suggests the Geometrical conception of an angle, which may be regarded as formed by the divergence of two straight lines from a point. In the definition of an angle, the magnitude of the angle is independent of the lengths of the two lines by which it is included; their mutual divergence from the point at which they meet, is the criterion of the magnitude of an angle, as it is pointed out in the succeeding definitions. The point at which the two lines meet is called the angular point or the vertex of the angle, and must not be confounded with the magnitude of the angle itself. The right angle is fixed in magnitude, and, on this account, it is made the standard with which all other angles are compared.

Two straight lines which actually intersect one another, or which when produced would intersect, are said to be inclined to one another, and the inclination of the two lines is determined by the angle which they make with one another.

Def. X. It may be here observed that in the Elements, Euclid always assumes that when one line is perpendicular to another line, the latter is also perpendicular to the former; and always calls *a right angle, ὀρθὴ γωνία*; but *a straight line, εὐθεῖα γραμμή*.

Def. XIX. This has been restored from Proclus, as it seems to have a meaning in the construction of Prop. 14, Book II; the first case of Prop. 33, Book III, and Prop. 13, Book VI. The definition of the segment of a circle is not once alluded to in Book I, and is not required before the discussion of the properties of the circle in Book III. Proclus remarks on this definition: "Hence you may collect that the center has three places: for it is either within the figure, as in the circle; or in its perimeter, as in the semicircle; or without the figure, as in certain conic lines."

Def. XXIV-XXIX. Triangles are divided into three classes, by reference to the relations of their sides; and into three other classes, by reference to their angles. A further classification may be made by considering both the relation of the sides and angles in each triangle.

In Simson's definition of the isosceles triangle, the word *only* must be omitted, as in the Cor. Prop. 5, Book I, an isosceles triangle may be equilateral, and an equilateral triangle is considered isosceles in Prop. 15, Book IV. Objection has been made to the definition of an acute-angled triangle. It is said that it cannot be admitted as a definition, that all the three angles of a triangle are acute, which is supposed in Def. 29. It may be replied, that the definitions of the three kinds of angles point out and seem to supply a foundation for a similar distinction of triangles.

Def. XXX-XXXIV. The definitions of quadrilateral figures are liable to objection. All of them, except the trapezium, fall under the general idea of a parallelogram; but as Euclid defined parallel straight lines after he had defined four-sided figures, no other arrangement could be adopted than the one he has followed; and for which there appeared to him, without doubt, some probable reasons. Sir Henry Savile, in his Seventh Lecture, remarks on some of the definitions of Euclid, "Nec dissimulandum aliquot harum in manibus exiguum esse usum in Geometriâ." A few verbal emendations have been made in some of them.

A square is a four-sided plane figure having all its sides equal, and one angle a right angle: because it is proved in Prop. 46, Book I, that if a *parallelogram* have one angle a right angle, all its angles are right *angles*.

An oblong, in the same manner, may be defined as a plane figure of four sides, having only its opposite sides equal, and one of its angles a right angle.

A rhomboid is a four-sided plane figure having only its opposite sides equal to one another and its angles not right angles.

Sometimes an irregular four-sided figure which has two sides parallel, is called a trapezoid.

Def. XXXV. It is possible for two right lines never to meet when produced, and not be parallel.

Def. A. The term parallelogram literally implies a figure formed by parallel straight lines, and may consist of four, six, eight, or any even number of sides, where every two of the opposite sides are parallel to one another. In the Elements, however, the term is restricted to four-sided figures, and includes the four species of figures named in the Definitions XXX—XXXIII.

The synthetic method is followed by Euclid not only in the demonstrations of the propositions, but also in laying down the definitions. He commences with the simplest abstractions, defining a point, a line, an angle, a superficies, and their different varieties. This mode of proceeding involves the difficulty, almost insurmountable, of defining satisfactorily the elementary abstractions of Geometry. It has been observed, that it is necessary to consider a solid, that is, a magnitude which has length, breadth, and thickness, in order to understand aright the definitions of a point, a line, and a superficies. A solid or volume considered apart from its physical properties, suggests the idea of the surfaces by which it is bounded: a surface, the idea of the line or lines which form its boundaries: and a finite line, the points which form its extremities. A solid is therefore bounded by surfaces; a surface is bounded by lines; and a line is terminated by two points. A point marks position only: a line has one dimension, length only, and defines distance: a superficies has two dimensions, length and breadth, and defines extension: and a solid has three dimensions, length, breadth, and thickness, and defines some portion of space.

It may also be remarked that two points are sufficient to determine the position of a straight line, and three points not in the same straight line, are necessary to fix the position of a plane.

ON THE POSTULATES.

THE definitions assume the possible existence of straight lines and circles, and the postulates predicate the possibility of drawing and of producing straight lines, and of describing circles. The postulates form the principles of construction assumed in the Elements; and are, in fact, problems, the possibility of which is admitted to be self-evident, and to require no proof.

It must, however, be carefully remarked, that the third postulate only admits that when any line is given in position and magnitude, a circle may be described from either extremity of the line as a center, and with a radius equal to the length of the line, as in Euc. I, 1. It does not admit the description of a circle with any other point as a center than one of the extremities of the given line.

Euc. I. 2, shews how, from any given point, to draw a straight line equal to another straight line which is given in magnitude and position.

ON THE AXIOMS.

Axioms are usually defined to be self-evident truths, which cannot be rendered more evident by demonstration; in other words, the axioms of Geometry are theorems, the truth of which is admitted without proof. It is by experience we first become acquainted with the different forms of geometrical magnitudes, and the axioms, or the fundamental ideas of their equality or inequality appear to rest on the same basis. The conception of the truth of the axioms does not appear to be more removed from experience than the conception of the definitions.

These axioms, or first principles of demonstration, are such theorems as cannot be resolved into simpler theorems, and no theorem ought to be admitted as a first principle of reasoning which is capable of being demonstrated. An axiom, and (when it is convertible) its converse, should both be of such a nature as that neither of them should require a formal demonstration.

The first and most simple idea, derived from experience is, that every magnitude fills a certain space, and that several magnitudes may successively fill the same space.

All the knowledge we have of magnitude is purely relative, and the most simple relations are those of equality and inequality. In the comparison of magnitudes, some are considered as given or known, and the unknown are compared with the known, and conclusions are synthetically deduced with respect to the equality or inequality of the magnitudes under consideration. In this manner we form our idea of equality, which is thus formally stated in the eighth axiom: "Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another."

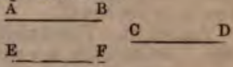
Every specific definition is referred to this universal principle. With regard to a few more general definitions which do not furnish an equality, it will be found that some hypothesis is always made reducing them to that principle, before any theory is built upon them. As for example, the definition of a straight line is to be referred to the tenth axiom; the definition of a right angle to the eleventh axiom; and the definition of parallel straight lines to the twelfth axiom.

The eighth axiom is called the principle of superposition, or, the mental process by which one Geometrical magnitude may be conceived to be placed on another, so as exactly to coincide with it, in the parts which are made the subject of comparison. Thus, if one straight line be conceived to be placed upon another, so that their extremities are coincident, the two straight lines are equal. If the directions of two lines which include one angle, coincide with the directions of the two lines which contain another angle, where the points, from which the angles diverge, coincide, then the two angles are equal: the lengths of the lines not affecting in any way the magnitudes of the angles. When one plane figure is conceived to be placed upon another, so that the boundaries of one exactly coincide with the boundaries of the other, then the two plane figures are equal. It may also be remarked, that the converse of this proposition is not universally true, namely, that when two magnitudes are equal, they coincide with one another: since two magnitudes may be equal in area, as two parallelograms or two triangles, Euc. I. 35, 37; but their boundaries may not be equal: and, consequently, by *superposition*, the figures could not exactly coincide: all such figures, *however, having equal* areas, by a different arrangement of their parts, *may be made to co*incide exactly.

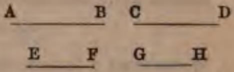
This axiom is the criterion of Geometrical equality, and is essentially different from the criterion of Arithmetical equality. Two geometrical magnitudes are equal, when they coincide or may be made to coincide: two abstract numbers are equal, when they contain the same aggregate of units; and two concrete numbers are equal, when they contain the same number of units of the same kind of magnitude. It is at once obvious, that Arithmetical representations of Geometrical magnitudes are not admissible in Euclid's criterion of Geometrical Equality, as he has not fixed the unit of magnitude of either the straight line, the angle, or the superficies. Perhaps Euclid intended that the first seven axioms should be applicable to numbers as well as to Geometrical magnitudes, and this is in accordance with the words of Proclus, who calls the axioms, *common notions*, not peculiar to the subject of Geometry.

Several of the axioms may be generally exemplified thus:

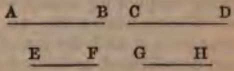
Axiom I. If the straight line AB be equal to the straight line CD; and if the straight line EF be also equal to the straight line CD; then the straight line AB is equal to the straight line EF.



Axiom II. If the line AB be equal to the line CD; and if the line EF be also equal to the line GH: then the sum of the lines AB and EF is equal to the sum of the lines CD and GH.

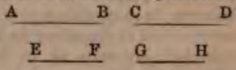


Axiom III. If the line AB be equal to the line CD; and if the line EF be also equal to the line GH; then the difference of AB and EF, is equal to the difference of CD and GH.

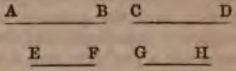


Axiom IV. admits of being exemplified under the two following forms:

1. If the line AB be equal to the line CD; and if the line EF *be greater than* the line GH; then the sum of the lines AB and EF *is greater than* the sum of the lines CD and GH.

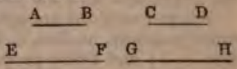


2. If the line AB be equal to the line CD; and if the line EF *be less than* the line GH; then the sum of the lines AB and EF *is less than* the sum of the lines CD and GH.

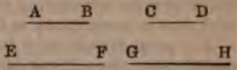


Axiom V. also admits of two forms of exemplification.

1. If the line AB be equal to the line CD; and if the line EF *be greater than* the line GH; then the difference of the lines AB and EF *is greater than* the difference of CD and GH.

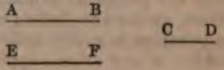


2. If the line AB be equal to the line CD; and if the line EF *be less than* the line GH; then the difference of the lines AB and EF *is less than* the difference of the lines CD and GH.

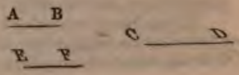


The axiom, "If unequals be taken from equals, the remainders are unequal," may be exemplified in the same manner.

Axiom VI. If the line AB be double of the line CD; and if the line EF be also double of the line CD; then the line AB is equal to the line EF.



Axiom VII. If the line AB be the half of the line CD; and if the line EF be also the half of the line CD; *then the line* AB is equal to the line EF.



It may be observed that when equal magnitudes are taken from unequal magnitudes, the greater remainder exceeds the less remainder by as much as the greater of the unequal magnitudes exceeds the less.

If unequals be taken from unequals, the remainders are not always unequal; they may be equal: also if unequals be added to unequals the wholes are not always unequal, they may also be equal.

Axiom IX. The whole is greater than its part, and conversely, the part is less than the whole. This axiom appears to assert the contrary of the eighth axiom, namely, that two magnitudes, of which one is greater than the other, cannot be made to coincide with one another.

Axiom X. The property of straight lines expressed by the tenth axiom, namely, "that two straight lines cannot enclose a space," is obviously implied in the definition of straight lines; for if they enclosed a space, they could not coincide between their extreme points, when the two lines are equal.

Axiom XI. This axiom has been asserted to be a demonstrable theorem. As an angle is a species of magnitude, this axiom is only a particular application of the eighth axiom to right angles.

Axiom XII. See the notes on Prop. XXIX. Book I.

ON THE PROPOSITIONS.

WHENEVER a judgment is formally expressed, there must be something respecting which the judgment is expressed, and something else which constitutes the judgment. The former is called the *subject* of the proposition, and the latter, the *predicate*, which may be anything which can be affirmed or denied respecting the *subject*.

The propositions in Euclid's Elements of Geometry may be divided into two classes, *problems* and *theorems*. A proposition, as the term imports, is something proposed; it is a *problem*, when some Geometrical *construction* is required to be *effected:* and it is a *theorem* when some Geometrical *property* is to be *demonstrated*. Every proposition is naturally divided into two parts; a problem consists of the *data*, or *things given;* and the *quæsita*, or *things required:* a theorem, consists of the *subject* or *hypothesis*, and the *conclusion*, or *predicate*. Hence the distinction between a problem and a theorem is this, that a problem consists of the data and the quæsita, and requires solution: and a theorem consists of the hypothesis and the predicate, and requires demonstration.

All propositions are *affirmative* or *negative;* that is, they either assert some property, as Euc. I. 4, or deny the existence of some property, as Euc. I. 7; and every proposition which is affirmatively stated has a contradictory corresponding proposition. If the affirmative be proved to be true, the contradictory is false.

All propositions may be viewed as (1) *universally affirmative*, or *universally negative*; (2) as *particularly affirmative*, or *particularly negative*.

The connected course of reasoning by which any Geometrical truth is established is called a demonstration. It is called a *direct* demonstration when the predicate of the proposition is inferred directly from the premisses, as the conclusion of a series of successive deductions. The demonstration is called *indirect*, when the conclusion shows that the introduction of any other supposition contrary to the hypothesis stated in the proposition, necessarily leads to an absurdity.

It has been remarked by Pascal, that "Geometry is almost the only *subject as to which* we find truths wherein all men agree; and one cause *of this is, that Geometers* alone regard the true laws of demonstration."

These are enumerated by him as eight in number. "1. To define nothing which cannot be expressed in clearer terms than those in which it is already expressed. 2. To leave no obscure or equivocal terms undefined. 3. To employ in the definition no terms not already known. 4. To omit nothing in the principles from which we argue, unless we are sure it is granted. 5. To lay down no axiom which is not perfectly evident. 6. To demonstrate nothing which is as clear already as we can make it. 7. To prove every thing in the least doubtful by means of self-evident axioms, or of propositions already demonstrated. 8. To substitute mentally the definition instead of the thing defined." Of these rules, he says, "the first, fourth and sixth are not absolutely necessary to avoid error, but the other five are indispensable; and though they may be found in books of logic, none but the Geometers have paid any regard to them."

The course pursued in the demonstrations of the propositions in Euclid's Elements of Geometry, is always to refer directly to some expressed principle, to leave nothing to be inferred from vague expressions, and to make every step of the demonstrations the object of the understanding.

It has been maintained by some philosophers, that a genuine definition contains some property or properties which can form a basis for demonstration, and that the science of Geometry is deduced from the definitions, and that on them alone the demonstrations depend. Others have maintained that a definition explains only the meaning of a term, and does not embrace the nature and properties of the thing defined.

If the propositions usually called postulates and axioms are either tacitly assumed or expressly stated in the definitions; in this view, demonstrations may be said to be legitimately founded on definitions. If, on the other hand, a definition is simply an explanation of the meaning of a term, whether abstract or concrete, by such marks as may prevent a misconception of the thing defined; it will be at once obvious that some constructive and theoretic principles must be assumed, besides the definitions to form the ground of legitimate demonstration. These principles we conceive to be the postulates and axioms. The postulates describe constructions which may be admitted as possible by direct appeal to our experience; and the axioms assert general theoretic truths so simple and self-evident as to require no proof, but to be admitted as the assumed first principles of demonstration. Under this view all Geometrical reasonings proceed upon the admission of the hypotheses assumed in the definitions, and the unquestioned possibility of the postulates, and the truth of the axioms.

Deductive reasoning is generally delivered in the form of an enthymeme, or an argument wherein one enunciation is not expressed, but is readily supplied by the reader: and it may be observed, that although this is the ordinary mode of speaking and writing, it is not in the strictly syllogistic form; as either the *major* or the *minor* premiss only is formally stated before the conclusion: Thus in Euc. I. 1.

Because the point A is the center of the circle BCD;
therefore the straight line AB is equal to the straight line AC.

The premiss here omitted, is: all straight lines drawn from the center of a circle to the circumference are equal.

In a similar way may be supplied the reserved premiss in every enthymeme. The conclusion of two enthymemes may form the major and minor premiss of a third syllogism, and so on, and thus any process of reasoning is reduced to the strictly syllogistic form. And in this way it is shewn that the general theorems of Geometry are demonstrated by means of syllogisms founded on the axioms and definitions.

Every syllogism consists of three propositions, of which, two are called the premisses, and the third, the conclusion. These propositions contain three terms, the subject and predicate of the conclusion, and the middle term which connects the predicate and the conclusion together. The subject of the conclusion is called *the minor*, and the predicate of the conclusion is called *the major* term, of the syllogism. The major term appears in one premiss, and the minor term in the other, with the middle term, which is in both premisses. That premiss which contains the middle term and the major term, is called the *major premiss;* and that which contains the middle term and the minor term, is called the *minor premiss* of the syllogism. As an example, we may take the syllogism in the demonstration of Prop. 1, Book 1, wherein it will be seen that the middle term is the subject of the major premiss and the predicate of the minor.

Major premiss: because the straight line AB is equal to the straight line AC;
Minor premiss: and, because the straight line BC is equal to the straight line AB;
Conclusion: therefore the straight line BC is equal to the straight line AC.

Here, BC is the subject, and AC the predicate of the conclusion.
BC is the subject, and AB the predicate of the minor premiss.
AB is the subject, and AC the predicate of the major premiss.

Also, AC is the major term, BC the minor term, and AB the middle term of the syllogism.

In this syllogism, it may be remarked that the definition of a straight line is assumed, and the definition of the Geometrical equality of two straight lines; also that a general theoretic truth, or axiom, forms the ground of the conclusion. And further, though it be impossible to make any point, mark or sign (σημεῖον) which has not both length and breadth, and any line which has not both length and breadth; the demonstrations in Geometry do not on this account become invalid. For they are pursued on the hypothesis that the point has no parts, but position only: and the line has length only, but no breadth or thickness: also that the surface has length and breadth only, but no thickness: and all the conclusions at which we arrive are independent of every other consideration.

The truth of the conclusion in the syllogism depends upon the truth of the premisses. If the premisses, or only one of them be not true, the conclusion is false. The conclusion is said *to follow from* the premisses; whereas, in truth, it is *contained in* the premisses. The expression must be understood of the mind apprehending in succession, the truth of the premisses, and subsequent to that, the truth of the conclusion; so that the conclusion *follows from* the premisses in order of time as far as reference is made to the mind's apprehension of the whole argument.

Every proposition, when complete, may be divided into six parts, as Proclus has pointed out in his commentary.

1. *The proposition*, or *general enunciation*, which states in general terms the conditions of the problem or theorem.

2. *The exposition*, or *particular enunciation*, which exhibits the *subject* of the proposition in particular terms as a fact, and refers it to some diagram described.

3. *The determination* contains the *predicate* in particular terms, as it is pointed out in the diagram, and directs attention to the demonstration, *by pronouncing the thing* sought.

4. *The construction* applies the postulates to prepare the diagram for the demonstration.

5. *The demonstration* is the connexion of syllogisms, which prove the truth or falsehood of the theorem, the possibility or impossibility of the problem, in that particular case exhibited in the diagram.

6. *The conclusion* is merely the repetition of the general enunciation, wherein the predicate is asserted as a demonstrated truth.

Prop. I. In the first two Books, the circle is employed as a mechanical instrument, in the same manner as the straight line, and the use made of it rests entirely on the third postulate. No properties of the circle are discussed in these books beyond the definition and the third postulate. When two circles are described, one of which has its center in the circumference of the other, the two circles being each of them partly within and partly without the other, their circumferences must intersect each other in two points; and it is obvious from the two circles cutting each other, in two points, one on each side of the given line, that two equilateral triangles may be formed on the given line.

Prop. II. When the given point is neither in the line, nor in the line produced, this problem admits of eight different lines being drawn from the given point in different directions, every one of which is a solution of the problem. For, 1. The given line has two extremities, to each of which a line may be drawn from the given point. 2. The equilateral triangle may be described on either side of this line. 3. And the side *BD* of the equilateral triangle *ABD* may be produced either way.

But when the given point lies either in the line or in the line produced, the distinction which arises from joining the two ends of the line with the given point, no longer exists, and there are only four cases of the problem.

The construction of this problem assumes a neater form, by first describing the circle *CGH* with center *B* and radius *BC*, and producing *DB* the side of the equilateral triangle *DBA* to meet the circumference in *G*: next, with center *D* and radius *DG*, describing the circle *GKL*, and then producing *DA* to meet the circumference in *L*.

By a similar construction the less of two given straight lines may be produced, so that the less together with the part produced may be equal to the greater.

Prop. III. This problem admits of two solutions, and it is left undetermined from which end of the greater line the part is to be cut off.

By means of this problem, a straight line may be found equal to the sum or the difference of two given lines.

Prop. IV. This forms the first case of equal triangles, two other cases are proved in Prop. VIII. and Prop. XXVI.

The term *base* is obviously taken from the idea of a building, and the same may be said of the term *altitude*. In Geometry, however, these terms are not restricted to one particular position of a figure, as in the case of a building, but may be in any position whatever.

Prop. V. Proclus has given, in his commentary, a proof for the equality of the angles at the base, without producing the equal sides. The construction follows the same order, taking in *AB* one side of the isosceles triangle *ABC*, a point *D* and cutting off from *AC* a part *AE* equal to *AD*, and then joining *CD* and *BE*.

A corollary is a theorem which results from the demonstration of a proposition.

Prop. VI. is the converse of one part of Prop. V. One proposition is defined to be the *converse* of another when the hypothesis of the former becomes the predicate of the latter; and vice versa.

There is besides this, another kind of conversion, when a theorem has several hypotheses and one predicate; by assuming the predicate and one, or more than one of the hypotheses, some one of the hypotheses may be inferred as the predicate of the converse. In this manner, Prop. VIII. is the converse of Prop. IV. It may here be observed, that converse theorems are not universally true: as for instance, the following direct proposition is universally true; "If two triangles have their three sides respectively equal, the three angles of each shall be respectively equal." But the converse is not universally true; namely, "If two triangles have the three angles in each respectively equal, the three sides are respectively equal." Converse theorems require, in some instances, the consideration of other conditions than those which enter into the proof of the direct theorem. *Converse* and *contrary* propositions are by no means to be confounded; the *contrary* proposition denies what is asserted, or asserts what is denied, in the *direct* proposition, but the subject and predicate in each are the same. A *contrary proposition* is a *completely contradictory proposition*, and the distinction consists in this—that *two contrary propositions* may both be false, but of *two contradictory propositions*, one of them must be true, and the other false. It may here be remarked, that one of the most common intellectual mistakes of learners, is to imagine that the denial of a proposition is a legitimate ground for affirming the contrary as true: whereas the rules of sound reasoning allow that the affirmation of a proposition as true, only affords a ground for the denial of the contrary as false.

Prop. VI. is the first instance of indirect demonstrations, and they are more suited for the proof of converse propositions. All those propositions which are demonstrated ex absurdo, are properly analytical demonstrations, according to the Greek notion of analysis, which first supposed the thing required, to be done, or to be true, and then shewed the consistency or inconsistency of this construction or hypothesis with truths admitted or already demonstrated.

In indirect demonstrations, where hypotheses are made which are not true and contrary to the truth stated in the proposition, it seems desirable that a form of expression should be employed different from that in which the hypotheses are true. In all cases therefore, whether noted by Euclid or not, the words *if possible* have been introduced, or some such qualifying expression, as in Euc. I. 6, so as not to leave upon the mind of the learner, the impression that the hypothesis which contradicts the proposition, is really true.

Prop. VIII. When the three sides of one triangle are shewn to coincide with the three sides of any other, the equality of the triangles is at once obvious. This, however, is not stated at the conclusion of Prop. VIII. or of Prop. XXVI. For the equality of the areas of two coincident triangles, reference is always made by Euclid to Prop. IV.

A direct demonstration may be given of this proposition, and Prop. VII. may be dispensed with altogether.

Let the triangles ABC, DEF be so placed that the base BC may coincide with the base EF, and the vertices A, D may be on opposite sides of EF. Join AD. Then because EAD is an isosceles triangle, the angle EAD is equal to the angle EDA; and because CDA is an isosceles triangle, the angle CAD is equal to the angle CDA. Hence the angle *EAF* is equal to the angle *EDF*, (ax. 2 or 3): or the angle *BDC* is equal to the angle *EDF*.

Prop. IX. If *BA*, *AC* be in the same straight line. This problem then becomes the same as Prob. XI, which may be regarded as drawing a line which bisects an angle equal to two right angles.

If *FA* be produced in the fig. Prop. 9, it bisects the angle which is the defect of the angle *BAC from four right angles.*

By means of this problem, any angle may be divided into four, eight, sixteen, &c. equal angles.

Prop. X. A finite straight line may, by this problem, be divided into four, eight, sixteen, &c. equal parts.

Prop. XI. When the point is at the extremity of the line; by the second postulate the line may be produced, and then the construction applies. See note on Euc. III. 31.

The distance between two points is the straight line which joins the points; but the distance between a point and a straight line, is *the shortest line* which can be drawn from the point to the line.

From this Prop. it follows that only one perpendicular can be drawn from a given point to a given line; and this perpendicular may be shewn to be less than any other line which can be drawn from the given point to the given line; and of the rest, the line which is nearer to the perpendicular is less than one more remote from it: also only two equal straight lines can be drawn from the same point to the line, one on each side of the perpendicular or the least. This property is analogous to Euc. III. 7, 8.

The corollary to this proposition is not in the Greek text, but was added by Simson, who states that it "is necessary to Prop. 1, Book XI., and otherwise."

Prop. XII. The third postulate requires that the line *CD* should be drawn before the circle can be described with the center *C*, and radius *CD*.

Prop. XIV. is the converse of Prop. XIII. "Upon the opposite sides of it." If these words were omitted, it is possible for two lines to make with a third, two angles, which together are equal to two right angles, in such a manner that the two lines shall not be in the same straight line.

The line *BE* may be supposed to fall above, as in Euclid's figure, or below the line *BD*, and the demonstration is the same in form.

Prop. XV. is the development of the definition of an angle. If the lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines, but in a different position.

The converse of this Proposition is not proved by Euclid, namely:— If the vertical angles made by four straight lines at the same point be respectively equal to each other, each pair of opposite lines shall be in the same straight line.

Prop. XVII. appears to be only a corollary to the preceding proposition, and it seems to be introduced to explain Axiom XII, of which it is the converse The exact truth respecting the angles of a triangle is proved in Prop. XXXII.

Prop. XVIII. It may here be remarked, for the purpose of guarding the student against a very common mistake, that in this proposition and in the converse of it, the *hypothesis* is stated before the *predicate*.

Prop. XIX. is the converse of Prop. XVIII. It may be remarked, that Prop. XIX. bears the same relation to Prop. XVIII., as Prop. VI. does to Prop. V.

Prop. xx. The following corollary arises from this proposition:—

A straight line is the shortest distance between two points. For the straight line *BC* is always less than *BA* and *AC*, however near the point *A* may be to the line *BC*.

It may be easily shewn from this proposition, that the difference of any two sides of a triangle is less than the third side.

Prop. xxii. When the sum of two of the lines is equal to, and when it is less than, the third line; let the diagrams be described, and they will exhibit the impossibility implied by the restriction laid down in the Proposition.

The same remark may be made here, as was made under the first Proposition, namely:—if one circle lies partly within and partly without another circle, the circumferences of the circles intersect each other in two points.

Prop. xxiii. *CD* might be taken equal to *CE*, and the construction effected by means of an isosceles triangle. It would, however, be less general than Euclid's, but is more convenient in practice.

Prop. xxiv. Simson makes the angle *EDG* at *D* in the line *ED*, the side which is not the greater of the two *ED*, *DF*; otherwise, three different cases would arise, as may be seen by forming the different figures. The point *G* might fall below or upon the base *EF* produced as well as above it. Prop. xxiv. and Prop. xxv. bear to each other the same relation as Prop. iv. and Prop. viii.

Prop. xxvi. This forms the third case of the equality of two triangles. Every triangle has three sides and three angles, and when any three of one triangle are given equal to any three of another, the triangles may be proved to be equal to one another, whenever the three magnitudes given in the hypothesis are independent of one another. Prop. iv. contains the first case, when the hypothesis consists of two sides and the included angle of each triangle. Prop. viii. contains the second, when the hypothesis consists of the three sides of each triangle. Prop. xxvi. contains the third, when the hypothesis consists of two angles, and one side either adjacent to the equal angles, or opposite to one of the equal angles in each triangle. There is another case, not proved by Euclid, when the hypothesis consists of two sides and one angle in each triangle, but these not the angles included by the two given sides in each triangle. This case however is only true under a certain restriction, thus:

If two triangles have two sides of one of them equal to two sides of the other, each to each, and have also the angles opposite to one of the equal sides in each triangle, equal to one another, and if the angles opposite to the other equal sides be both acute, or both obtuse angles; then shall the third sides be equal in each triangle, as also the remaining angles of the one to the remaining angles of the other.

Let *ABC*, *DEF* be two triangles which have the sides *AB*, *AC* equal to the two sides *DE*, *DF*, each to each, and the angle *ABC* equal to the angle *DEF*: then, if the angles *ACB*, *DEF*, be *both acute*, or *both obtuse angles*, the third side *BC* shall be equal to the third side *EF*, and also the angle *BCA* to the angle *EFD*, and the angle *BAC* to the angle *EDF*.

First. Let the angles *ACB*, *DFE* opposite to the equal sides *AB*, *DE*, be *both acute angles*.

If BC be not equal to *EF*, let *BC* be the greater, and from *BC*, cut off *BG equal to EF*, and join *AG*.

Then in the triangles ABG, *DEF*, Euc. i. 4. AG is equal to DF, and the angle *AGB* to *DFE*. But since *AC* is equal to *DF*, *AG* is equal to *AC*: and therefore the angle *ACG* is equal to the angle *AGC*, which is also an acute angle. But because *AGC*, *AGB* are together equal to two right angles, and that *AGC* is an acute angle, *AGB* must be an obtuse angle; which is absurd. Wherefore, *BC* is not unequal to *EF*, that is, *BC* is equal to *EF*, and also the remaining angles of one triangle to the remaining angles of the other.

Secondly. Let the angles *ACB*, *DFE*, be both *obtuse angles*. By proceeding in a similar way, it may be shewn that *BC* cannot be otherwise than equal to *EF*.

If *ACB*, *DFE* be *both right angles:* the case falls under Euc. I. 26.

Prop. xxvii. Alternate angles are defined to be the two angles which two straight lines make with another at its extremities, but upon opposite sides of it.

When a straight line intersects two other straight lines, two pairs of alternate angles are formed by the lines at their intersections, as in the figure, *BEF*, *EFC* are alternate angles as well as the angles *AEF*, *EFD*.

Prop. xxviii. One angle is called "the exterior angle," and another "the interior and opposite angle," when they are formed on the same side of a straight line which falls upon or intersects two other straight lines. It is also obvious that on each side of the line, there will be two exterior and two interior and opposite angles. The exterior angle *EGB* has the angle *GHD* for its corresponding interior and opposite angle: also the exterior angle *FHD* has the angle *HGB* for its interior and opposite angle.

Prop. xxix is the converse of Prop. xxvii and Prop. xxviii.

As the definition of parallel straight lines simply describes them by a statement of the negative property, that they never meet; it is necessary that some positive property of parallel lines should be assumed as an axiom, on which reasonings on such lines may be founded.

Euclid has assumed the statement in the twelfth axiom, which has been objected to, as not being self-evident. A stronger objection appears to be, that the converse of it forms Euc. I. 17; for both the assumed axiom and its converse, should be so obvious as not to require formal demonstration.

Simson has attempted to overcome the objection, not by any improved definition and axiom respecting parallel lines; but, by considering Euclid's twelfth axiom to be a theorem, and for its proof, assuming two definitions and one axiom, and then demonstrating five subsidiary Propositions.

Instead of Euclid's twelfth axiom, the following has been proposed as a more simple property for the foundation of reasonings on parallel lines; namely, "If a straight line fall on two parallel straight lines, the alternate angles are equal to one another." In whatever this may exceed Euclid's definition in simplicity, it is liable to a similar objection, being the converse of Euc. I. 27.

Professor Playfair has adopted in his Elements of Geometry, that "Two straight lines which intersect one another cannot be both parallel to the same straight line." This apparently more simple axiom follows as a direct inference from Euc. I. 30.

But one of the least objectionable of all the definitions which have been proposed on this subject, appears to be that which simply expresses the conception of equidistance. It may be formally stated thus: "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other." This includes the conception stated by Euclid, that parallel lines never meet. Dr. Wallis observes on this subject, "Parallelismus et æquidistantia vel idem sunt, vel certe se mutuo comitantur."

As an additional reason for this definition being preferred, it may be remarked that the meaning of the terms γραμμαὶ παράλληλοι, suggests the exact idea of such lines.

An account of thirty methods which have been proposed at different times for avoiding the difficulty in the twelfth axiom, will be found in the appendix to Colonel Thompson's "Geometry without Axioms."

Prop. xxx. In the diagram, the two lines *AB* and *CD* are placed one on each side of the line *EF*: the proposition may also be proved when both *AB* and *CD* are on the same side of *EF*.

Prop. xxxii. From this proposition, it is obvious that if one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle, as is shewn in Euc. iii. 31, and that each of the angles of an equilateral triangle, is equal to two thirds of a right angle, as it is shewn in Euc. iv. 15. Also, if one angle of an isosceles triangle be a right angle, then each of the equal angles is half a right angle, as in Euc. ii. 9.

The three angles of a triangle may be shewn to be equal to two right angles without producing a side of the triangle, by drawing through any angle of the triangle a line parallel to the opposite side, as Proclus has remarked in his Commentary on this proposition. It is manifest from this proposition, that the third angle of a triangle is not independent of the sum of the other two; but is known if the sum of any two is known. Cor. 1 may be also proved by drawing lines from any one of the angles of the figure to the other angles. If any of the sides of the figure bend inwards and form what are called re-entering angles, the enunciation of these two corollaries will require some modification. As Euclid gives no definition of re-entering angles, it may fairly be concluded, he did not intend to enter into the proofs of the properties of figures which contain such angles.

Prop. xxxiii. The words "towards the same parts" are a necessary restriction: for if they were omitted, it would be doubtful whether the extremities *A*, *C*, and *B*, *D* were to be joined by the lines *AC* and *BD*; or the extremities *A*, *D*, and *B*, *C*, by the lines *AD* and *BC*.

Prop. xxxiv. If the other diameter be drawn, it may be shewn that the diameters of a parallelogram bisect each other, as well as bisect the area of the parallelogram. If the parallelogram be right angled, the diagonals are equal; if the parallelogram be a square or a rhombus, the diagonals bisect each other at right angles. The converse of this Prop., namely, "If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall also be parallel; that is, the figure shall be a parallelogram," is not proved by Euclid.

Prop. xxxv. The latter part of the demonstration is not expressed very intelligibly. Simson, who altered the demonstration, seems in fact to consider two trapeziums of the same form and magnitude, and from one of them, to take the triangle *ABE*; and from the other, the triangle *DCF*; and then the remainders are equal by the third axiom: that is, the parallelogram *ABCD* is equal to the parallelogram *EBCF*. Otherwise, the triangle, whose base is *DE*, (fig. 2.) is taken twice from the trapezium, which would appear to be impossible, if the sense in which Euclid applies the third axiom, is to be retained here.

It may be observed, that the two parallelograms exhibited in fig. 2 partially lie on one another, and that the triangle whose base is *BC* is a common part of them, but that the triangle whose base is *DE* is entirely without both the parallelograms. After having proved the triangle *ABE* equal to the triangle *DCF*, if we take from these equals (fig. 2.) the triangle whose base is *DE*, and to each of the remainders add the triangle whose base is *BC*, then the parallelogram *ABCD* is equal to the parallelogram *EBCF*. In fig. 3, the equality of the parallelograms *ABCD*, *EBCF*, is shewn by adding the figure *EBCD* to each of the triangles *ABE*, *DCF*.

In this proposition, the word *equal* assumes a new meaning, and is no longer restricted to mean coincidence in all the parts of two figures.

Prop. XXXVIII. In this proposition, it is to be understood that the bases of the two triangles are in the same straight line. If in the diagram the point *E* coincide with *C*, and *D* with *A*, then the angle of one triangle is supplemental to the other. Hence the following property:—If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles supplemental, the two triangles are equal.

A distinction ought to be made between *equal* triangles and *equivalent* triangles, the former including those whose sides and angles mutually coincide, the latter those whose areas only are equivalent.

Prop. XXXIX. If the vertices of all the equal triangles which can be described upon the same base, or upon the equal bases as in Prop. 40, be joined, the line thus formed will be a straight line, and is called the locus of the vertices of equal triangles upon the same base, or upon equal bases.

A locus in plane Geometry is a straight line or a plane curve, every point of which and none else satisfies a certain condition. With the exception of the straight line and the circle, the two most simple loci; all other loci, perhaps including also the Conic Sections, may be more readily and effectually investigated algebraically by means of their rectangular or polar equations.

Prop. XLI. The converse of this proposition is not proved by Euclid; viz. If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line, and towards the same parts, they shall be between the same parallels. Also, it may easily be shewn that if two equal triangles are between the same parallels; they are either upon the same base, or upon equal bases.

Prop. XLIV. A parallelogram described on a straight line is said to be *applied* to that line.

Prop. XLV. The problem is solved only for a rectilineal figure of four sides. If the given rectilineal figure have more than four sides, it may be divided into triangles by drawing straight lines from any angle of the figure to the opposite angles, and then a parallelogram equal to the third triangle can be applied to *LM*, and having an angle equal to *E*: and so on for all the triangles of which the rectilineal figure is composed.

Prop. XLVI. The square being considered as an equilateral rectangle, its area or surface may be expressed numerically if the number of lineal units in a side of the square be given, as is shewn in the note on Prop. I., Book II.

The student will not fail to remark the analogy which exists between the *area of a square* and *the product of two equal numbers*; and between *the side of a square* and *the square foot of a number*. There is, however, this distinction to be observed; it is always possible to find the product of two equal numbers, (or *to find the square of a number*, as it is usually called,) and to describe a square on a given line; but conversely, though the side of a given square is known from the figure itself, the exact number of units in the side of a square of given area, can only be found exactly, in such cases where the given number is a square number. For example, if the area of a square contain 9 square units, then the square root of 9 or 3, indicates the number of lineal units in the side of that square. Again, if the area of a square contain 12 square units, the side of the square is greater than 3, but less than 4 lineal units, and there is no number which will exactly express the side of that square: an approximation to the true length, however, may be obtained to any assigned degree of accuracy.

Prop. XLVII. In a right-angled triangle, the side opposite to the right angle is called the hypotenuse, and the other two sides, the base and perpendicular, according to their position.

In the diagram the three squares are described on the *outer* sides of the triangle *ABC*. The Proposition may also be demonstrated (1) when the three squares are described upon the *inner* sides of the triangle: (2) when one square is described on the outer side and the other two squares on the inner sides of the triangle: (3) when one square is described on the inner side and the other two squares on the outer sides of the triangle.

As one instance of the third case. If the square *BE* on the hypotenuse be described on the inner side of *BC* and the squares *BG*, *HC* on the outer sides of *AB*, *AC*; the point *D* falls on the side *FG* (Euclid's fig.) of the square *BG*, and *KH* produced meets *CE* in *E*. Let *LA* meet *BC* in *M*. Join *DA*; then the square *GB* and the oblong *LB* are each double of the triangle *DAB*, (Euc. I. 41.); and similarly by joining *EA*, the square *HC* and oblong *LC* are each double of the triangle *EAC*. Whence it follows that the squares on the sides *AB*, *AC* are together equal to the square on the hypotenuse *BC*.

By this proposition may be found a square equal to the sum of any given squares, or equal to any multiple of a given square: or equal to the difference of two given squares.

The truth of this proposition may be exhibited to the eye in some particular instances. As in the case of that right-angled triangle whose three sides are 3, 4, and 5 units respectively. If through the points of division of two contiguous sides of each of the squares upon the sides, lines be drawn parallel to the sides (see the notes on Book II.), it will be obvious, that the squares will be divided into 9, 16 and 25 small squares, each of the same magnitude; and that the number of the small squares into which the squares on the perpendicular and base are divided is equal to the number into which the square on the hypotenuse is divided.

Prop. XLVIII is the converse of Prop. XLVII. In this Prop. is assumed the Corollary that "the squares described upon two equal lines are equal," and the converse, which properly ought to have been appended to Prop. XLVI.

The First Book of Euclid's Elements, it has been seen, is conversant with the construction and properties of rectilineal figures. It first lays down the definitions which limit the subjects of discussion in the First Book, next the three postulates, which restrict the instruments by which the constructions in Plane Geometry are effected; and thirdly, the twelve axioms, which express the principles by which a comparison is made between the ideas of the things defined.

This Book may be divided into three parts. The first part treats of the origin and properties of triangles, both with respect to their sides and angles; and the comparison of these mutually, both with regard to equality and inequality. The second part treats of the properties of parallel lines and of parallelograms. The third part exhibits the connection of the properties of triangles and parallelograms, and the equality of the squares on the base and perpendicular of a right-angled triangle to the square on the hypotenuse.

When the propositions of the First Book have been read with the notes, the student is recommended to use different letters in the diagrams, and where it is possible, diagrams of a form somewhat different from those exhibited in the text, for the purpose of testing the accuracy of his knowledge of the demonstrations. And further, when he has become sufficiently familiar with the method of geometrical reasoning, he may dispense with the aid of letters altogether, and acquire the power of expressing in general terms the process of reasoning in the demonstration of any proposition. Also, he is advised to answer the following questions before he attempts to apply the principles of the First Book to the solution of Problems and the demonstration of Theorems.

QUESTIONS ON BOOK I.

1. What is the name of the Science of which Euclid gives the Elements? What is meant by *Solid Geometry?* Is there any distinction between *Plane Geometry*, and the *Geometry of Planes?*

2. Define the term *magnitude*, and specify the different kinds of magnitude considered in Geometry. What dimensions of space belong to figures treated of in the first six Books of Euclid?

3. Give Euclid's definition of a "straight line." What does he really use as his test of rectilinearity, and where does he first employ it? What objections have been made to it, and what substitute has been proposed as an available definition? How many points are necessary to fix the position of a straight line in a plane? When is one straight line said to *cut*, and when to *meet* another?

4. What positive property has a Geometrical point? From the definition of a straight line, shew that the intersection of two lines is a point.

5. Give Euclid's definition of a plane rectilineal angle. What are the limits of the angles considered in Geometry? Does Euclid consider angles greater than two right angles?

6. When is a straight line said to be drawn at *right angles*, and when *perpendicular*, to a given straight line?

7. Define a *triangle;* shew how many kinds of triangles there are according to the variation both of the *angles*, and of the *sides*.

8. What is Euclid's definition of a circle? Point out the assumption involved in your definition. Is any axiom implied in it? Shew that in this as in all other definitions some geometrical fact is assumed as somehow previously known.

9. Define the quadrilateral figures mentioned by Euclid.

10. Describe briefly the use and foundation of definitions, axioms, and postulates: give illustrations by an instance of each.

11. What objection may be made to the method and order in which Euclid has laid down the elementary abstractions of the Science of Geometry? What other method has been suggested?

12. What distinctions may be made between definitions in the Science of Geometry and in the Physical Sciences?

13. What is necessary to constitute an exact definition? Are definitions propositions? Are they arbitrary? Are they convertible? Does a Mathematical definition admit of proof on the principles of the Science to which it relates?

14. Enumerate the principles of construction assumed by Euclid.

15. Of what instruments may the use be considered to meet approximately the demands of Euclid's postulates? Why only *approximately?*

16. "A circle may be described from any center, with any straight line as radius." How does this postulate differ from Euclid's, and which of his problems is assumed in it?

17. What principles in the Physical Sciences correspond to axioms in Geometry?

18. Enumerate Euclid's twelve axioms and point out those which have special reference to Geometry. State the converse of those which admit of being so expressed.

19. What two tests of equality are assumed by Euclid? Is the assumption of the principle of superposition (ax. 8.), essential to all Geometrical reasoning? Is it correct to say, that it is "an appeal, though of the most familiar sort, to external observation"?

20. Could any, and if any, which of the axioms of Euclid be turned into definitions; and with what advantages or disadvantages?

21. Define the terms, Problem, Postulate, Axiom and Theorem. Are any of Euclid's axioms improperly so called?

22. Of what two parts does the enunciation of a Problem, and of a Theorem consist? Distinguish them in Euc. I. 4, 5, 18, 19.

23. When is a problem said to be indeterminate? Give an example.

24. When is one proposition said to be the converse or reciprocal of another? Give examples. Are converse propositions universally true? If not, under what circumstances are they necessarily true? Why is it necessary to demonstrate converse propositions? How are they proved?

25. Explain the meaning of the word *proposition*. Distinguish between *converse* and *contrary* propositions, and give examples.

26. State the grounds as to whether Geometrical reasonings depend for their conclusiveness upon axioms or definitions.

27. Explain the meaning of *enthymeme* and *syllogism*. How is the enthymeme made to assume the form of the syllogism? Give examples.

28. What constitutes a demonstration? State the laws of demonstration.

29. What are the principle parts, in the entire process of establishing a proposition?

30. Distinguish between a *direct* and *indirect* demonstration.

31. What is meant by the term *synthesis*, and what, by the term, *analysis?* Which of these modes of reasoning does Euclid adopt in his Elements of Geometry?

32. In what sense is it true that the conclusions of Geometry are necessary truths?

33. Enunciate those Geometrical definitions which are used in the proof of the propositions of the First Book.

34. If in Euclid I. 1, an equal triangle be described on the other side of the given line, what figure will the two triangles form?

35. In the diagram, Euclid I. 2, if DB a side of the equilateral triangle DAB be produced both ways and cut the circle whose center is B and radius BC in two points G and H; shew that either of the dis-

tances *DG*, *DH* may be taken as the radius of the second circle; and give the proof in each case.

36. Explain how the propositions Euc. I. 2, 3, are rendered necessary by the restriction imposed by the third postulate. Is it necessary for the proof, that the triangle described in Euc. I. 2, should be equilateral? Could we, at this stage of the subject, describe an isosceles triangle on a given base?

37. State how Euc. I. 2, may be extended to the following problem: "From a given point to draw a straight line *in a given direction* equal to a given straight line."

38. How would you cut off from a straight line unlimited in both directions, a length equal to a given straight line?

39. In the proof of Euclid I. 4, how much depends upon Definition, how much upon Axiom?

40. Draw the figure for the third case of Euc. I. 7, and state why it *needs no demonstration.*

41. In the construction Euclid I. 9, is it indifferent in all cases on which side of the joining line the equilateral triangle is described?

42. Shew how a given straight line may be bisected by Euc. I. 1.

43. In what cases do the lines which bisect the interior angles of plane triangles, also bisect one, or more than one of the corresponding opposite sides of the triangles?

44. "Two straight lines cannot have a common segment." Has this corollary been tacitly assumed in any preceding proposition?

45. In Euc. I. 12, must the given line necessarily be "of unlimited length"?

46. Shew that (fig. Euc. I. 11) every point without the perpendicular drawn from the middle point of every straight line *DE*, is at unequal distances from the extremities *D*, *E* of that line.

47. From what proposition may it be inferred that a straight line is the shortest distance between two points?

48. Enunciate the propositions you employ in the proof of Euc. I. 16.

49. Is it essential to the truth of Euc. I. 21, that the two straight lines be drawn from the extremities of the base?

50. In the diagram, Euc. I. 21, by how much does the greater angle *BDC* exceed the less *BAC*?

51. To form a triangle with three straight lines, any two of them must be greater than the third: is a similar limitation necessary with respect to the three angles?

52. Is it possible to form a triangle with three lines whose lengths are 1, 2, 3 units: or one with three lines whose lengths are 1, $\sqrt{2}$, $\sqrt{3}$?

53. Is it possible to construct a triangle whose angles shall be as the numbers 1, 2, 3? Prove or disprove your answer.

54. What is the reason of the limitation in the construction of Euc. I. 24. viz. "that *DE* is that side which is not greater than the other?"

55. Quote the first proposition in which the equality of two areas which cannot be superposed on each other is considered.

56. Is the following proposition universally true? "If two plane triangles have three elements of the one respectively equal to three elements of the other, the triangles are equal in every respect." Enumerate all the cases in which this equality is proved in the First Book. What case is omitted?

57. What parts of a triangle must be given in order that the triangle may be described?

58. State the converse of the second case of Euc. I. 26? Under what limitations is it true? Prove the proposition so limited?

59. Shew that the angle contained between the perpendiculars drawn to two given straight lines which meet each other, is equal to the angle contained by the lines themselves.

60. Are two triangles necessarily equal in all respects, where a side and two angles of the one are equal to a side and two angles of the other, each to each?

61. Illustrate fully the difference between analytical and synthetical proofs. What propositions in Euclid are demonstrated analytically?

62. Can it be properly predicated of any two straight lines that they never meet if indefinitely produced either way, antecedently to our knowledge of some other property of such lines, which makes the property first predicated of them a necessary conclusion from it?

63. Enunciate Euclid's definition and axiom relating to parallel straight lines; and state in what Props. of Book I. they are used.

64. What proposition is the converse to the twelfth axiom of the First Book? What other two propositions are complementary to these?

65. If lines being produced ever so far do not meet; can they be otherwise than parallel? If so, under what circumstances?

66. Define *adjacent angles*, *opposite angles*, *vertical angles*, and *alternate angles*; and give examples from the First Book of Euclid.

67. Can you suggest anything to justify the assumption in the twelfth axiom upon which the proof of Euc. I. 29, depends?

68. What objections have been urged against the definition and the doctrine of parallel straight lines as laid down by Euclid? Where does the difficulty originate? What other assumptions have been suggested and for what reasons?

69. Assuming as an axiom that two straight lines which cut one another cannot both be parallel to the same straight line; deduce Euclid's twelfth axiom as a corollary of Euc. I. 29.

70. From Euc. I. 27, shew that the distance between two parallel straight lines is constant?

71. If two straight lines be not parallel, shew that all straight lines falling on them, make alternate angles, which differ by the same angle.

72. Taking as the definition of parallel straight lines that they are equally inclined to the same straight line towards the same parts; prove that "being produced ever so far both ways they do not meet?" Prove also Euclid's axiom 12, by means of the same definition.

73. What is meant by *exterior* and *interior* angles? Point out examples.

74. Can the three angles of a triangle be proved equal to two right angles without producing a side of the triangle?

75. Shew how the corners of a triangular piece of paper may be turned down, so as to exhibit to the eye that the three angles of a triangle are equal to two right angles.

76. Explain the meaning of the term *corollary*. Enunciate the two corollaries appended to Euc. I. 32, and give another proof of the first What other corollaries may be deduced from this proposition?

77. Shew that the two lines which bisect the exterior and interior angles of a triangle, as well as those which bisect any two interior angles of a parallelogram, contain a right angle.

78. *The opposite sides and angles of a parallelogram are equal to one another, and the diameters bisect it.* State and prove the converse *of this proposition.* Also shew that a quadrilateral figure, is a paral-

lelogram, when its diagonals bisect each other; and when its diagonals divide it into four triangles, which are equal, two and two, viz. those which have the same vertical angles.

79. If two straight lines join the extremities of two parallel straight lines, but *not* towards the same parts, when are the joining lines equal, and when are they unequal?

80. If either diameter of a four-sided figure divide it into two equal triangles, is the figure necessarily a parallelogram? Prove your answer.

81. Shew how to divide one of the parallelograms in Euc. I. 35, by straight lines so that the parts when properly arranged shall make up the other parallelogram.

82. Distinguish between *equal* triangles and *equivalent* triangles, and give examples from the First Book of Euclid.

83. What is meant by the locus of a point? Adduce instances of loci from the first Book of Euclid.

84. How is it shewn that equal triangles upon the same base or equal bases have equal altitudes, whether they are situated on the same or opposite sides of the same straight line?

85. In Euc. I. 37, 38, if the triangles are not towards the same parts, shew that the straight line joining the vertices of the triangles is bisected by the line containing the bases.

86. If the complements (fig. Euc. I. 43) be squares, determine their relation to the whole parallelogram.

87. What is meant by a parallelogram being applied to a straight line?

88. Is the proof of Euc. I. 45, perfectly general?

89. Define a square without including superfluous conditions, and explain the mode of constructing a square upon a given straight line in conformity with such a definition.

90. The sum of the angles of a square is equal to four right angles. Is the converse true? If not, why?

91. Conceiving a square to be a figure bounded by four equal straight lines not necessarily in the same plane, what condition respecting the angles is necessary to complete the definition?

92. In Euclid I. 47, why is it necessary to prove that one side of each square described upon each of the sides containing the right angle, should be in the same straight line with the other side of the triangle?

93. On what assumption is an analogy shewn to exist between the product of two equal numbers and the surface of a square?

94. Is the triangle whose sides are 3, 4, 5 right-angled, or not?

95. Can the side and diagonal of a square be represented simultaneously by any finite numbers?

96. By means of Euc. I. 47, the square roots of the natural numbers, 1, 2, 3, 4, &c. may be represented by straight lines.

97. If the square on the hypotenuse in the fig. Euc. I. 47, be described on the other side of it: shew from the diagram how the squares on the two sides of the triangle may be made to cover exactly the square on the hypotenuse.

98. If Euclid II. 2, be assumed, enunciate the form in which Euc. I. 47 may be expressed.

99. Classify all the properties of *triangles* and *parallelograms*, proved in the First Book of Euclid.

100. Mention any propositions in Book I. which are included in more general ones which follow.

ON THE ANCIENT GEOMETRICAL ANALYSIS.

SYNTHESIS, or the method of composition, is a mode of reasoning which begins with something given, and ends with something required, either to be done or to be proved. This may be termed a *direct process*, as it leads from principles to consequences.

Analysis, or the method of resolution, is the reverse of synthesis, and thus it may be considered an *indirect process*, a method of reasoning from consequences to principles.

The synthetic method is pursued by Euclid in his Elements of Geometry. He commences with certain assumed principles, and proceeds to the solution of problems and the demonstration of theorems by undeniable and successive inferences from them.

The Geometrical Analysis was a process employed by the ancient Geometers, both for the discovery of the solution of problems and for the investigation of the truth of theorems. In the analysis of a *problem*, the quæsita, or what is required to be done, is supposed to have been effected, and the consequences are traced by a series of geometrical constructions and reasonings, till at length they terminate in the data of the problem, or in some previously demonstrated or admitted truth, whence the direct solution of the problem is deduced.

In the Synthesis of a *problem*, however, the last consequence of the analysis is assumed as the first step of the process, and by proceeding in a contrary order through the several steps of the analysis until the process terminate in the quæsita, the solution of the problem is effected.

But if, in the analysis, we arrive at a consequence which contradicts any truth demonstrated in the Elements, or which is inconsistent with the data of the problem, the problem must be impossible: and further, if in certain relations of the given magnitudes the construction be possible, while in other relations it is impossible, the discovery of these relations will become a necessary part of the solution of the problem.

In the analysis of a *theorem*, the question to be determined, is, whether by the application of the geometrical truths proved in the Elements, the predicate is consistent with the hypothesis. This point is ascertained by assuming the predicate to be true, and by deducing the successive consequences of this assumption combined with proved geometrical truths, till they terminate in the hypothesis of the theorem or some demonstrated truth. The theorem will be proved synthetically by retracing, in order, the steps of the investigation pursued in the analysis, till they terminate in the predicate, which was assumed in the analysis. This process will constitute the demonstration of the theorem.

If the assumption of the truth of the predicate in the analysis lead to some consequence which is inconsistent with any demonstrated truth, the false conclusion thus arrived at, indicates the falsehood of the predicate; and by reversing the process of the analysis, it may be demonstrated, that the theorem cannot be true.

It may here be remarked, that the geometrical analysis is more extensively useful in discovering the solution of problems than for investigating the demonstration of theorems.

From the nature of the subject, it must be at once obvious, that no general rules can be prescribed, which will be found applicable in all cases, and infallibly lead to the solution of every problem. The conditions of problems must suggest what constructions may be possible; and the consequences which follow from these constructions and the assumed solution, will shew the possibility or impossibility of arriving at some known property consistent with the data of the problem.

Though the data of a problem may be given in magnitude and position, certain ambiguities will arise, if they are not properly restricted. Two points may be considered as situated on the same side, or one on each side of a given line; and there may be two lines drawn from a given point making equal angles with a line given in position; and to avoid ambiguity, it must be stated on which side of the line the angle is to be formed.

A problem is said to be *determinate* when, with the prescribed conditions, it admits of one definite solution; the same construction which may be made on the other side of any given line, not being considered a different solution: and a problem is said to be *indeterminate* when it admits of more than one definite solution. This latter circumstance arises from the data not *absolutely fixing*, but *merely restricting* the quæsita, leaving certain points or lines not fixed in one position only. The number of given conditions may be insufficient for a single determinate solution; or relations may subsist among some of the given conditions from which one or more of the remaining given conditions may be deduced.

If the base of a right-angled triangle be given, and also the difference of the squares on the hypotenuse and perpendicular, the triangle is indeterminate. For though apparently here are three things given, the right angle, the base, and the difference of the squares on the hypotenuse and perpendicular, it is obvious that these three apparent conditions are in fact reducible to two: for since in a right-angled triangle, the sum of the squares on the base and on the perpendicular, is equal to the square on the hypotenuse, it follows that the difference of the squares on the hypotenuse and perpendicular, is equal to the square on the base of the triangle, and therefore the base is known from the difference of the squares on the hypotenuse and perpendicular being known. The conditions therefore are insufficient to determine a right-angled triangle; an indefinite number of triangles may be found with the prescribed conditions, whose vertices will lie in the line which is perpendicular to the base.

If a problem relate to the determination of *a single point*, and the data be sufficient to determine the position of that point, the problem is *determinate:* but if one or more of the conditions be omitted, the data which remain may be sufficient for the determination of more than one point, each of which satisfies the conditions of the problem; in that case, the problem is *indeterminate:* and in general, such points are found to be situated in some line, and hence such line is called the locus of the point which satisfies the conditions of the problem.

If any two given points A and B (fig. Euc. IV. 5.) be joined by a straight *line* AB, and this line be bisected in D, then if a *perpendicular* be drawn from the point of bisection, it is manifest that a circle described with *any* point in the perpendicular as a center, and a radius equal to its distance from one of the given points, will pass through the other point, and the perpendicular will be the locus of all the circles which can be described passing through the two given points.

Again, if a third point *C* be taken, but not in the same straight line with the other two, and this point be joined with the first point *A*; then the perpendicular drawn from the bisection *E* of this line will be the locus of the centers of all circles which pass through the first and third points *A* and *C*. But the perpendicular at the bisection of the first and second points *A* and *B* is the locus of the centers of circles which pass through these two points. Hence the intersection *F* of these two perpendiculars, will be the center of a circle which passes through the three points and is called the intersection of the two loci. Sometimes this method of solving geometrical problems may be pursued with advantage, by constructing the locus of every two points separately, which are given in the conditions of the problem. In the Geometrical Exercises which follow, only those local problems are given where the locus is either a straight line or a circle.

Whenever the quæsitum is a point, the problem on being rendered indeterminate, becomes a locus, whether the deficient datum be of the essential or of the accidental kind. When the quæsitum is a straight line or a circle, (which were the only two loci admitted into the ancient Elementary Geometry) the problem *may* admit of an *accidentally indeterminate* case; but will not *invariably* or even very frequently do so. This will be the case, when the line or circle shall be so far arbitrary in its position, as depends upon the deficiency of a *single* condition to fix it perfectly ;—that is, (for instance) one point in the line, or two points in the circle, may be determined from the given conditions, but the remaining one is indeterminate from the accidental relations among the data of the problem.

Determinate Problems become indeterminate by the merging of some one datum in the results of the remaining ones. This may arise in three different ways; first, from the coincidence of two points; secondly, from that of two straight lines; and thirdly, from that of two circles. These, further, are the only three ways in which this accidental coincidence of data can produce this indeterminateness; that is, in other words, convert the problem into a Porism.

In the original Greek of Euclid's Elements, the corollaries to the propositions are called porisms (πορισματα); but this scarcely explains the nature of *porisms*, as it is manifest that they are. different from simple deductions from the demonstrations of propositions. Some analogy, however, we may suppose them to have to the porisms or corollaries in the Elements. Pappus (Coll. Math. Lib. VII. pref.) informs us that Euclid wrote three books on Porisms. He defines "a porism to be something between a problem and a theorem, or that in which something is proposed to be investigated." Dr. Simson, to whom is due the merit of having restored the porisms of Euclid, gives the following definition of that class of propositions: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem, datis, sed quæ ad ea quæ data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam." That is, "A Porism is a proposition in which it is proposed to demonstrate that some one thing, or more things than one, are given, to which, as also to each of innumerable other things, not given indeed, but which have the same relation to those which are given, it is to be shewn that there belongs some common affection described in the proposition." Professor Dugald Stewart defines a porism to be "A proposition affirming the possibility of finding one or more of the conditions of an indeterminate theorem." Professor Playfair in a paper (from which the following account is taken) on Porisms, printed in the Transactions of the Royal Society of Edinburgh, for the year 1792, defines a porism to be "A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of innumerable solutions."

It may without much difficulty be perceived that this definition represents a porism as almost the same as an indeterminate problem. There is a large class of indeterminate problems which are, in general, loci, and satisfy certain defined conditions. Every indeterminate problem containing a locus may be made to assume the form of a porism, but not the converse. Porisms are of a more general nature than indeterminate problems which involve a locus.

The ancient geometers appear to have undertaken the solution of problems with a scrupulous and minute attention, which would scarcely allow any of the collateral truths to escape their observation. They never considered a problem as solved till they had distinguished all its varieties, and evolved separately every different case that could occur, carefully distinguishing whatever change might arise in the construction from any change that was supposed to take place among the magnitudes which were given. This cautious method of proceeding soon led them to see that there were circumstances in which the solution of a problem would cease to be possible; and this always happened when one of the conditions of the data was inconsistent with the rest. Such instances would occur in the simplest problems; but in the analysis of more complex problems, they must have remarked that their constructions failed, for a reason directly contrary to that assigned. Instances would be found where the lines, which, by their intersection, were to determine the thing sought, instead of intersecting one another, as they did in general, or of not meeting at all, would coincide with one another entirely, and consequently leave the question unresolved. The confusion thus arising would soon be cleared up, by observing, that a problem before determined by the intersection of two lines, would now become capable of an indefinite number of solutions. This was soon perceived to arise from one of the conditions of the problem involving another, or from two parts of the data becoming one, so that there was not left a sufficient number of independent conditions to confine the problem to a single solution, or any determinate number of solutions. It was not difficult afterwards to perceive, that these cases of problems formed very curious propositions, of an indeterminate nature between problems and theorems, and that they admitted of being enunciated separately. It was to such propositions so enunciated that the ancient geometers gave the name of Porisms.

Besides, it will be found, that some problems are possible withi certain limits, and that certain magnitudes increase while others decrease within those limits; and after having reached a certain value, the former begin to decrease, while the latter increase. This circumstance gives rise to questions of *maxima* and *minima*, or the greatest and least values which certain magnitudes may admit of in indeterminate problems.

In the following collection of problems and theorems, most will be found to be of so simple a character, (being almost obvious deductions from propositions in the Elements) as scarcely to admit of the principle of the Geometrical Analysis being applied, in their solution.

It must however be recollected that a clear and exact knowledge of the first principles of Geometry must necessarily precede any intelligent application of them. Indistinctness or defectiveness of understanding with respect to these, will be a perpetual source of error and confusion. The learner is therefore recommended to understand the principles of the Science, and their connexion, fully, before he attempt any applications of them. The following directions may assist him in his proceedings.

ANALYSIS OF THEOREMS.

1. Assume that the Theorem is true.
2. Proceed to examine any consequences that result from this admission, by the aid of other truths respecting the diagram, which have been already proved.
3. Examine whether any of these consequences are already known to be *true*, or to be *false*.
4. If any one of them be false, we have arrived at a *reductio ad absurdum*, which proves that the theorem itself is false, as in Euc. I. 25.
5. If none of the consequences so deduced be *known* to be either true or false, proceed to deduce other consequences from all or any of these, as in (2).
6. Examine these results, and proceed as in (3) and (4); and if still without any conclusive indications of the truth or falsehood of the alleged theorem, proceed still further, until such are obtained.

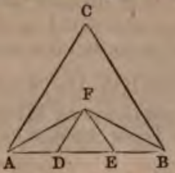
ANALYSIS OF PROBLEMS.

1. In general, any given problem will be found to depend on several problems and theorems, and these ultimately on some problem or theorem in Euclid.
2. Describe the diagram as directed in the enunciation, and suppose the solution of the problem effected.
3. Examine the relations of the lines, angles, triangles, &c. in the diagram, and find the dependence of the assumed solution on some theorem or problem in the Elements.
4. If such cannot be found, draw other lines parallel or perpendicular as the case may require, join given points, or points assumed in the solution, and describe circles if need be: and then proceed to *trace the dependence* of the assumed solution on some theorem or *problem in Euclid*.
5. *Let not the first* unsuccessful attempts at the solution of a *Problem be considered* as of no value; such attempts have been found *to lead to the discovery* of other theorems and problems.

PROPOSITION I. PROBLEM.

To trisect a given straight line.

ANALYSIS. Let AB be the given straight line, and suppose it divided into three equal parts in the points D, E.



On DE describe an equilateral triangle DEF,
then DF is equal to AD, and FE to EB.
On AB describe an equilateral triangle ABC,
and join AF, FB.
Then because AD is equal to DF,
therefore the angle AFD is equal to the angle DAF,
and the two angles DAF, DFA are double of one of them DAF.
But the angle FDE is equal to the angles DAF, DFA,
and the angle FDE is equal to DAC, each being an angle of an equilateral triangle;
therefore the angle DAC is double the angle DAF;
wherefore the angle DAC is bisected by AF.
Also because the angle FAC is equal to the angle FAD,
and the angle FAD to DFA;
therefore the angle CAF is equal to the alternate angle AFD:
and consequently FD is parallel to AC.

Synthesis. Upon AB describe an equilateral triangle ABC,
bisect the angles at A and B by the straight lines AF, BF, meeting in F;
through F draw FD parallel to AC, and FE parallel to BC.
Then AB is trisected in the points D, E.
For since AC is parallel to FD and FA meets them,
therefore the alternate angles FAC, AFD are equal;
but the angle FAD is equal to the angle FAC,
hence the angle DAF is equal to the angle AFD,
and therefore DF is equal to DA.
But the angle FDE is equal to the angle CAB,
and FED to CBA; (I. 29.)
therefore the remaining angle DFE is equal to the remaining angle ACB.
Hence the three sides of the triangle DFE are equal to one another,
and DF has been shewn to be equal to DA,
therefore AD, DE, EB are equal to one another.

Hence the following theorem.

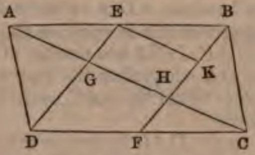
If the angles at the base of an equilateral triangle be bisected by two lines which meet at a point within the triangle; the two lines drawn from this point parallel to the sides of the triangle, divide the base into three equal parts.

Note. There is another method whereby a line may be divided into three equal parts:—by drawing from one extremity of the given line, another making an acute angle with it, and taking three equal distances from the extremity, then joining the extremities, and through the other two points of division, drawing lines parallel to this line through the other two points of division, and to the given line; the three triangles thus formed are equal in all respects. This may be extended for any number of parts, and is a particular case of Euc. VI. 10.

PROPOSITION II. THEOREM.

If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal.

Let $ABCD$ be a parallelogram of which the diagonal is AC.
Let AB be bisected in E, and DC in F,
also let DE, FB be joined cutting the diagonal in G, H.
Then AC is trisected in the points G, H.



Through E draw EK parallel to AC and meeting FB in K.
Then because EB is the half of AB, and DF the half of DC;
therefore EB is equal to DF;
and these equal and parallel straight lines are joined towards the same parts by DE and FB;
therefore DE and FB are equal and parallel. (I. 33.)
And because AEB meets the parallels EK, AC,
therefore the exterior angle BEK is equal to the interior angle EAG.
For a similar reason, the angle EBK is equal to the angle AEG.

Hence in the triangles AEG, EBK, there are the two angles GAE, AEG in the one, equal to the two angles KEB, EBK in the other, and one side adjacent to the equal angles in each triangle, namely AE equal to EB;

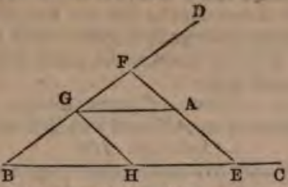
therefore AG is equal to EK, (I. 26.)
but EK is equal to GH, (I. 34.) therefore AG is equal to GH.
By a similar process, it may be shewn that GH is equal to HC.
Hence AG, GH, HC are equal to one another,
and therefore AC is trisected in the points G, H.
It may also be proved that BF is trisected in H and K.

PROPOSITION III. PROBLEM.

Draw through a given point, between two straight lines not parallel, a straight line which shall be bisected in that point.

Analysis. Let BC, BD be the two lines meeting in B, and let A be the given point between them.

Suppose the line EAF drawn through A, so that EA is equal to AF;



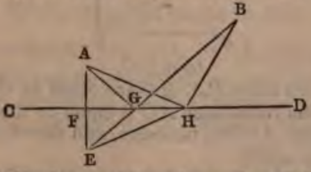
through A draw AG parallel to BC, and GH parallel to EF.
Then $AGHE$ is a parallelogram, wherefore AE is equal to GH,
but EA is equal to AF by hypothesis; therefore GH is equal to AF.
Hence in the triangles BHG, GAF,
the angles HBG, AGF are equal, as also BGH, GFA, (I. 29.)
also the side GH is equal to AF;
whence the other parts of the triangles are equal, (I. 26.)
therefore BG is equal to GF.

Synthesis. Through the given point A, draw AG parallel to BC;
on GD, take GF equal to GB;
then F is a second point in the required line:
join the points F, A, and produce FA to meet BC in E;
then the line FE is bisected in the point A;
draw GH parallel to AE.
Then in the triangles BGH, GFA, the side BG is equal to GF,
and the angles GBH, BGH are respectively equal to FGA, GFA;
wherefore GH is equal to AF, (I. 26.)
but GH is equal to AE, (I. 34.)
therefore AE is equal to AF, or EF is bisected in A.

PROPOSITION IV. PROBLEM.

From two given points on the same side of a straight line given in position, draw two straight lines which shall meet in that line, and make equal angles with it; also prove, that the sum of these two lines is less than the sum of any other two lines drawn to any other point in the line.

Analysis. Let A, B be the two given points, and CD the given line.
Suppose G the required point in the line, such that AG and BG being joined, the angle AGC is equal to the angle BGD.



Draw AF perpendicular to CD and meeting BG produced in E.
Then, because the angle BGD is equal to AGF, (hyp.)
and also to the vertical angle FGE, (I. 15.)
therefore the angle AGF is equal to the angle EGF;

also the right angle *AFG* is equal to the right angle *EFG*,
and the side *FG* is common to the two triangles *AFG*, *EFG*,
therefore *AG* is equal to *EG*, and *AF* to *FE*.

Hence the point *E* being known, the point *G* is determined by the intersection of *CD* and *BE*.

Synthesis. From *A* draw *AF* perpendicular to *CD*, and produce it to *E*, making *FE* equal to *AF*, and join *BE* cutting *CD* in *G*.

Join also *AG*.

Then *AG* and *BG* make equal angles with *CD*.

For since *AF* is equal to *FE*, and *FG* is common to the two triangles *AGF*, *EGF*, and the included angles *AFG*, *EFG* are equal;
therefore the base *AG* is equal to the base *EG*,
and the angle *AGF* to the angle *EGF*;
but the angle *EGF* is equal to the vertical angle *BGD*,
therefore the angle *AGF* is equal to the angle *BGD*;
that is, the straight lines *AG* and *BG* make equal angles with the straight line *CD*.

Also the sum of the lines *AG*, *GB* is a minimum.

For take any other point *H* in *CD*, and join *EH*, *HB*, *AH*.
Then since any two sides of a triangle are greater than the third side,
therefore *EH*, *HB* are greater than *EB* in the triangle *EHB*.
But *EG* is equal to *AG*, and *EH* to *AH*;
therefore *AH*, *HB* are greater than *AG*, *GB*.

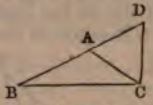
That is, *AG*, *GB* are less than any other two lines which can be drawn from *A*, *B*, to any other point *H* in the line *CD*.

By means of this Proposition may be found the shortest path from one given point to another, subject to the condition, that it shall meet two given lines.

PROPOSITION V. PROBLEM.

Given one angle, a side opposite to it, and the sum of the other two sides, construct the triangle.

Analysis. Suppose *BAC* the triangle required, having *BC* equal to the given side, *BAC* equal to the given angle opposite to *BC*, also *BD* equal to the sum of the other two sides.



Join *DC*.

Then since the two sides *BA*, *AC* are equal to *BD*, by taking *BA* from these equals, the remainder *AC* is equal to the remainder *AD*.

Hence the triangle *ACD* is isosceles, and therefore the angle *ADC* is equal to the angle *ACD*.

But the exterior angle *BAC* of the triangle *ADC* is equal to the two interior and opposite angles *ACD* and *ADC*:

Wherefore the angle *BAC* is double the angle *BDC*, and *BDC* *is the half of the angle* *BAC*.

Hence the synthesis.

At the point D in BD, make the angle BDC equal to half the given angle,

and from B the other extremity of BD, draw BC equal to the given side, and meeting DC in C,

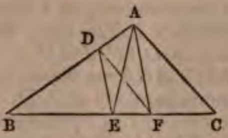
at C in CD make the angle DCA equal to the angle CDA, so that CA may meet BD in the point A.

Then the triangle ABC shall have the required conditions.

PROPOSITION VI. PROBLEM.

To bisect a triangle by a line drawn from a given point in one of the sides.

Analysis. Let ABC be the given triangle, and D the given point in the side AB.



Suppose DF the line drawn from D which bisects the triangle;
therefore the triangle DBF is half of the triangle ABC.
Bisect BC in E, and join AE, DE, AF,
then the triangle ABE is half of the triangle ABC:
hence the triangle ABE is equal to the triangle DBF;
take away from these equals the triangle DBE,
therefore the remainder ADE is equal to the remainder DEF.

But ADE, DEF are equal triangles upon the same base DE, and on the same side of it,
they are therefore between the same parallels, (I. 39.)
that is, AF is parallel to DE,
therefore the point F is determined.

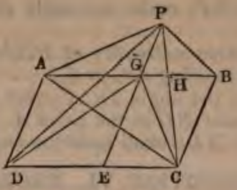
Synthesis. Bisect the base BC in E, join DE,
from A, draw AF parallel to DE, and join DF.
Then because DE is parallel to AF,
therefore the triangle ADE is equal to the triangle DEF; (I. 37.)
to each of these equals, add the triangle BDE,
therefore the whole triangle ABE is equal to the whole DBF,
but ABE is half of the whole triangle ABC;
therefore DBF is also half of the triangle ABC.

PROPOSITION VII. THEOREM.

If from a point without a parallelogram lines be drawn to the extremities of two adjacent sides, and of the diagonal which they include; of the triangles thus formed, that, whose base is the diagonal, is equal to the sum of the other two.

Let $ABCD$ be a parallelogram of which AC is one of the diagonals, and let P be any point without it: and let AP, PC, BP, PD be joined.

Then the triangles APD, APB are together equivalent to the triangle APC.



Draw PGE parallel to AD or BC, and meeting AB in G, and DC in E; and join DG, GC.

Then the triangles CBP, CBG are equal: (I. 37.)
and taking the common part CBH from each,
the remainders PHB, CHG are equal.
Again, the triangles DAP, DAG are equal; (I. 37.)
also the triangles DAG, AGC are equal, being on the same base AG, and between the same parallels AG, DC:
therefore the triangle DAP is equal to the triangle AGC:
but the triangle PHB is equal to the triangle CHG,
wherefore the triangles PHB, DAP are equal to AGC, CHG, or ACH, add to these equals the triangle APH,
therefore the triangles APH, PHB, DAP are equal to APH, ACH,
that is, the triangles APB, DAP are together equal to the triangle PAC.

If the point P be within the parallelogram, then the *difference* of the triangles APB, DAP may be proved to be equal to the triangle PAC.

I.

8. Describe an isosceles triangle upon a given base and having each of the sides double of the base, without using any proposition of the Elements subsequent to the first three. If the base and sides be given, what condition must be fulfilled with regard to the magnitude of each of the equal sides in order that an isosceles triangle may be constructed?

9. In the fig. Euc. I. 5. If FC and BG meet in H, then prove that AH bisects the angle BAC.

10. In the fig. Euc. I. 5. If the angle FBG be equal to the angle ABC, and BG, CF, intersect in O; the angle BOF is equal to twice the angle BAC.

11. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides, the angles made by them with the base are each equal to half the vertical angle.

12. A line drawn bisecting the angle contained by the two equal *sides of an isosceles* triangle, bisects the third side at right angles.

13. *If a straight* line drawn bisecting the vertical angle of a tr *angle also bisect* the base, the triangle is isosceles.

14. Given two points one on each side of a given straight line; find a point in the line such that the angle contained by two lines drawn to the given points may be bisected by the given line.

15. In the fig. Euc. I. 5, let F and G be the points in the sides AB and AC produced, and let lines FH and GK be drawn perpendicular and equal to FC and GB respectively: also if BH, CK, or these lines produced meet in O; prove that BH is equal to CK, and BO to CO.

16. From every point of a given straight line, the straight lines drawn to each of two given points on opposite sides of the line are equal: prove that the line joining the given points will cut the given line at right angles.

17. If A be the vertex of an isosceles triangle ABC, and BA be produced so that AD is equal to BA, and DC be drawn; shew that BCD is a right angle.

18. The straight line EDF, drawn at right angles to BC the base of an isosceles triangle ABC, cuts the side AB in D, and CA produced in E; shew that AED is an isosceles triangle.

19. In the fig. Euc. I. 1, if AB be produced both ways to meet the circles in D and E, and from C, CD and CE be drawn; the figure CDE is an isosceles triangle having each of the angles at the base, equal to one fourth of the angle at the vertex of the triangle.

20. From a given point, draw two straight lines making equal angles with two given straight lines intersecting one another.

21. From a given point to draw a straight line to a given straight line, that shall be bisected by another given straight line.

22. Place a straight line of given length between two given straight lines which meet, so that it shall be equally inclined to each of them.

23. To determine that point in a straight line from which the straight lines drawn to two other given points shall be equal, provided the line joining the two given points is not perpendicular to the given line.

24. In a given straight line to find a point equally distant from two given straight lines. In what case is this impossible?

25. If a line intercepted between the extremity of the base of an isosceles triangle, and the opposite side (produced if necessary) be equal to a side of the triangle, the angle formed by this line and the base produced, is equal to three times either of the equal angles of the triangle.

26. In the base BC of an isosceles triangle ABC, take a point D, and in CA take CE equal to CD, let ED produced meet AB produced in F; then $3.AEF = 2$ right angles $+ AFE$, or $= 4$ right angles $+ AFE$.

27. If from the base to the opposite sides of an isosceles triangle, three straight lines be drawn, making equal angles with the base, viz. one from its extremity, the other two from any other point in it, these two shall be together equal to the first.

28. A straight line is drawn, terminated by one of the sides of an isosceles triangle, and by the other side produced, and bisected by the base; prove that the straight lines, thus intercepted between the

vertex of the isosceles triangle, and this straight line, are together equal to the two equal sides of the triangle.

29. In a triangle, if the lines bisecting the angles at the base be equal, the triangle is isosceles, and the angle contained by the bisecting lines is equal to an exterior angle at the base of the triangle.

30. In a triangle, if lines be equal when drawn from the extremities of the base, (1) perpendicular to the sides, (2) bisecting the sides, (3) making equal angles with the sides; the triangle is isosceles: and then these lines which respectively join the intersections of the sides, are parallel to the base.

II.

31. *ABC* is a triangle right-angled at *B*, and having the angle *A* double the angle *C*; shew that the side *BC* is less than double the side *AB*.

32. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

33. If from the right angle of a right-angled triangle, two straight lines be drawn, one perpendicular to the base, and the other bisecting it, they will contain an angle equal to the difference of the two acute angles of the triangle.

34. If the vertical angle *CAB* of a triangle *ABC* be bisected by *AD*, to which the perpendiculars *CE*, *BF* are drawn from the remaining angles: bisect the base *BC* in *G*, join *GE*, *GF*, and prove these lines equal to each other.

35. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

36. If one angle at the base of a triangle be double of the other, the less side is equal to the sum or difference of the segments of the base made by the perpendicular from the vertex, according as the angle is greater or less than a right angle.

37. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle, it will bisect that angle.

38. From the vertex of a scalene triangle draw a right line to the base, which shall exceed the less side as much as it is exceeded by the greater.

39. Divide a right angle into three equal angles.

40. One of the acute angles of a right-angled triangle is three times as great as the other; trisect the smaller of these.

41. Prove that the sum of the distances of any point within a triangle from the three angles is greater than half the perimeter of the triangle.

42. The perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.

43. If from the angles of a triangle *ABC*, straight lines *ADE*, *BDF*, *CDG* be drawn through a point D to the opposite sides, prove that the sides of the triangle are together greater than the three

lines drawn to the point D, and less than twice the same, but greater than two-thirds of the lines drawn through the point to the opposite sides.

44. In a plane triangle an angle is right, acute or obtuse, according as the line joining the vertex of the angle with the middle point of the opposite side is equal to, greater or less than half of that side.

45. If the straight line AD bisect the angle A of the triangle ABC, and BDE be drawn perpendicular to AD and meeting AC or AC produced in E, shew that $BD = DE$.

46. The side BC of a triangle ABC is produced to a point D. The angle ACB is bisected by a line CE which meets AB in E. A line is drawn through E parallel to BC and meeting AC in F, and the line bisecting the exterior angle ACD, in G. Shew that EF is equal to FG.

47. The sides AB, AC, of a triangle are bisected in D and E respectively, and BE, CD, are produced until $EF = EB$, and $GD = DC$; shew that the line GF passes through A.

48. In a triangle ABC, AD being drawn perpendicular to the straight line BD which bisects the angle B, shew that a line drawn from D parallel to BC will bisect AC.

49. If the sides of a triangle be trisected and lines be drawn through the points of section adjacent to each angle so as to form another triangle, this shall be in all respects equal to the first triangle.

50. Between two given straight lines it is required to draw a straight line which shall be equal to one given straight line, and parallel to another.

51. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, another bisecting the base, and the third perpendicular to the base, the first is always intermediate in magnitude and position to the other two.

52. In the base of a triangle, find the point from which, lines drawn parallel to the sides of the triangle and limited by them, are equal.

53. In the base of a triangle, to find a point from which if two lines be drawn, (1) perpendicular, (2) parallel, to the two sides of the triangle, their sum shall be equal to a given line.

III.

54. In the figure of Euc. I. 1, the given line is produced to meet either of the circles in P; shew that P and the points of intersection of the circles, are the angular points of an equilateral triangle.

55. If each of the equal angles of an isosceles triangle be one-fourth of the third angle, and from one of them a line be drawn at right angles to the base meeting the opposite side produced; then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

56. In the figure Euc. I. 1, if the sides CA, CB of the equilateral triangle ABC be produced to meet the circles in F, G, respectively, and if C' be the point in which the circles cut one another on the

other side of AB: prove the points F, C', G to be in the same straight line; and the figure CFG to be an equilateral triangle.

57. ABC is a triangle and the exterior angles at B and C are bisected by lines BD, CD respectively, meeting in D: shew that the angle BDC and half the angle BAC make up a right angle.

58. If the exterior angle of a triangle be bisected, and the angles of the triangle made by the bisectors be bisected, and so on, the triangles so formed will tend to become eventually equilateral.

59. If in the three sides AB, BC, CA of an equilateral triangle ABC, distances AE, BF, CG be taken, each equal to a third of one of the sides, and the points E, F, G be respectively joined (1) with each other, (2) with the opposite angles: shew that the two triangles so formed, are equilateral triangles.

IV.

60. Describe a right-angled triangle upon a given base, having given also the perpendicular from the right angle upon the hypotenuse.

61. Given one side of a right-angled triangle, and the difference between the hypotenuse and the sum of the other two sides, to construct the triangle.

62. Construct an isosceles right-angled triangle, having given (1) the sum of the hypotenuse and one side; (2) their difference.

63. Describe a right-angled triangle of which the hypotenuse and the difference between the other two sides are given.

64. Given the base of an isosceles triangle, and the sum or difference of a side and the perpendicular from the vertex on the base. Construct the triangle.

65. Make an isosceles triangle of given altitude whose sides shall pass through two given points and have its base on a given straight line.

66. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the angles on the opposite side.

67. Having given the straight lines which bisect the angles at the base of an equilateral triangle, determine a side of the triangle.

68. Having given two sides and an angle of a triangle, construct the triangle, distinguishing the different cases.

69. Having given the base of a triangle, the difference of the sides, and the difference of the angles at the base; to describe the triangle.

70. Given the perimeter and the angles of a triangle, to construct it.

71. Having given the base of a triangle, and half the sum and half the difference of the angles at the base; to construct the triangle.

72. Having given two lines, which are not parallel, and a point between them; describe a triangle having two of its angles in the respective lines, and the third at the given point; and such that the sides shall be equally inclined to the lines which they meet.

73. Construct a triangle, having given the three lines drawn from the angles to bisect the sides opposite.

74. Given one of the angles at the base of a triangle, the base itself, and the sum of the two remaining sides, to construct the triangle.

75. Given the base, an angle adjacent to the base, and the difference of the sides of a triangle, to construct it.

76. Given one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.

77. Given the base and the sum of the two other sides of a triangle, construct it so that the line which bisects the vertical angle shall be parallel to a given line.

V.

78. From a given point without a given straight line, to draw a line making an angle with the given line equal to a given rectilineal angle.

79. Through a given point A, draw a straight line ABC meeting two given parallel straight lines in B and C, such that BC may be equal to a given straight line.

80. If the line joining two parallel lines be bisected, all the lines drawn through the point of bisection and terminated by the parallel lines are also bisected in that point.

81. Three given straight lines issue from a point: draw another straight line cutting them so that the two segments of it intercepted between them may be equal to one another.

82. AB, AC are two straight lines, B and C given points in the same; BD is drawn perpendicular to AC, and DE perpendicular to AB; in like manner CF is drawn perpendicular to AB, and FG to AC. Shew that EG is parallel to BC.

83. ABC is a right-angled triangle, and the sides AC, AB are produced to D and F; bisect FBC and BCD by the lines BE, CE, and from E let fall the perpendiculars EF, ED. Prove (without assuming any properties of parallels) that $ADEF$ is a square.

84. Two pairs of equal straight lines being given, shew how to construct with them the greatest parallelogram.

85. With two given lines as diagonals describe a parallelogram which shall have an angle equal to a given angle. Within what limits must the given angle lie?

86. Having given one of the diagonals of a parallelogram, the sum of the two adjacent sides and the angle between them, construct the parallelogram.

87. One of the diagonals of a parallelogram being given, and the angle which it makes with one of the sides, complete the parallelogram, so that the other diagonal may be parallel to a given line.

88. $ABCD$, $A'B'C'D'$ are two parallelograms whose corresponding sides are equal, but the angle A is greater than the angle A', prove that the diameter AC is less than $A'C'$, but BD greater than $B'D'$.

89. If in the diagonal of a parallelogram any two points equidistant from its extremities be joined with the opposite angles, a figure will be formed which is also a parallelogram.

90. From each angle of a parallelogram a line is drawn making

the same angle towards the same parts with an adjacent side, taken always in the same order; shew that these lines form another parallelogram *similar* to the original one.

91. Along the sides of a parallelogram taken in order, measure $AA' = BB' = CC' = DD'$: the figure $A'B'C'D'$ will be a parallelogram.

92. On the sides AB, BC, CD, DA, of a parallelogram, set off AE, BF, CG, DH, equal to each other, and join AF, BG, CH, DE: these lines form a parallelogram, and the difference of the angles AFB, BGC, equals the difference of any two proximate angles of the two parallelograms.

93. OB, OC are two straight lines at right angles to each other, through any point P any two straight lines are drawn intersecting OB, OC, in B, B', C, C', respectively. If D and D' be the middle points of BB' and CC', shew that the angle $B'PD'$ is equal to the angle DOD'.

94. $ABCD$ is a parallelogram of which the angle C is opposite to the angle A. If through A any straight line be drawn, then the distance of C is equal to the sum or difference of the distances of B and of D from that straight line, according as it lies without or within the parallelogram.

95. Upon stretching two chains AC, BD, across a field $ABCD$, I find that BD and AC make equal angles with DC, and that AC makes the same angle with AD that BD does with BC; hence prove that AB is parallel to CD.

96. To find a point in the side or side produced of any parallelogram, such that the angle it makes with the line joining the point and one extremity of the opposite side, may be bisected by the line joining it with the other extremity.

97. When the corner of the leaf of a book is turned down a second time, so that the lines of folding are parallel and equidistant, the space in the second fold is equal to three times that in the first.

VI.

98. If the points of bisection of the sides of a triangle be joined. the triangle so formed shall be one-fourth of the given triangle.

99. If in the triangle ABC, BC be bisected in D, AD joined and bisected in E, BE joined and bisected in F, and CF joined and bisected in G; then the triangle EFG will be equal to one-eighth of the triangle ABC.

100. Shew that the areas of the two equilateral triangles in Prob. 59, p. 78, are respectively, one-third and one-seventh of the area of the original triangle.

101. To describe a triangle equal to a given triangle, (1) when the base, (2) when the altitude of the required triangle is given.

102. To describe a triangle equal to the sum or difference of two given triangles.

103. Upon a given base describe an isosceles triangle equal to a given triangle.

104. Describe a right-angled triangle equal to a given triangle ABC.

105. To a given straight line apply a triangle which shall be equal

to a given parallelogram and have one of its angles equal to a given rectilineal angle.

106. Transform a given rectilineal figure into a triangle whose vertex shall be in a given angle of the figure, and whose base shall be in one of the sides.

107. Divide a triangle by two straight lines into three parts which when properly arranged shall form a parallelogram whose angles are of a given magnitude.

108. Shew that a scalene triangle cannot be divided into two parts which will coincide.

109. If two sides of a triangle be given, the triangle will be greatest when they contain a right angle.

110. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.

111. Of all triangles having the same base and the same perimeter, that is the greatest which has the two undetermined sides equal.

112. Divide a triangle into three equal parts, (1) by lines drawn from a point in one of the sides: (2) by lines drawn from the angles to a point within the triangle: (3) by lines drawn from a given point within the triangle. In how many ways can the third case be done?

113. Divide an equilateral triangle into nine equal parts.

114. Bisect a parallelogram, (1) by a line drawn from a point in one of its sides: (2) by a line drawn from a given point within or without it: (3) by a line perpendicular to one of the sides: (4) by a line drawn parallel to a given line.

115. From a given point in one side produced of a parallelogram, draw a straight line which shall divide the parallelogram into two equal parts.

116. To trisect a parallelogram by lines drawn (1) from a given point in one of its sides, (2) from one of its angular points.

VII.

117. To describe a rhombus which shall be equal to any given quadrilateral figure.

118. Describe a parallelogram which shall be equal in area and perimeter to a given triangle.

119. Find a point in the diagonal of a square produced, from which if a straight line be drawn parallel to any side of the square, and meeting another side produced, it will form together with the produced diagonal and produced side, a triangle equal to the square.

120. If from any point within a parallelogram, straight lines be drawn to the angles, the parallelogram shall be divided into four triangles, of which each two opposite are together equal to one-half of the parallelogram.

121. If $ABCD$ be a parallelogram, and E any point in the diagonal AC, or AC produced; shew that the triangles EBC, EDC, are equal, as also the triangles EBA and EBD.

122. $ABCD$ is a parallelogram, draw DFG meeting BC in F,

and AB produced in G; join AF, CG; then will the triangles ABF, CFG be equal to one another.

123. $ABCD$ is a parallelogram, E the point of intersection of its diagonals, and K any point in AD. If KB, KC be joined, shew that the figure $BKEC$ is one-fourth of the parallelogram.

124. Let $ABCD$ be a parallelogram, and O any point within it, through O draw lines parallel to the sides of $ABCD$, and join OA, OC; prove that the difference of the parallelograms DO, BO is twice the triangle OAC.

125. The diagonals AC, BD of a parallelogram intersect in O, and P is a point within the triangle AOB; prove that the difference of the triangles APB, CPD is equal to the sum of the triangles APC, BPD.

126. If K be the common angular point of the parallelograms about the diameter AC (fig. Euc. I. 43.) and BD be the other diameter, the difference of these parallelograms is equal to twice the triangle BKD.

127. The perimeter of a square is less than that of any other parallelogram of equal area.

128. Shew that of all equiangular parallelograms of equal perimeters, that which is equilateral is the greatest.

129. Prove that the perimeter of an isosceles triangle is greater than that of an equal right-angled parallelogram of the same altitude.

VIII.

130. If a quadrilateral figure is bisected by one diagonal, the second diagonal is bisected by the first.

131. If two opposite angles of a quadrilateral figure are equal, shew that the angles between opposite sides produced are equal.

132. Prove that the sides of any four-sided rectilinear figure are together greater than the two diagonals.

133. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles, from any point within the figure, except their intersection.

134. The longest side of a given quadrilateral is opposite to the shortest; shew that the angles adjacent to the shortest side are together greater than the sum of the angles adjacent to the longest side.

135. Give any two points in the opposite sides of a trapezium, inscribe in it a parallelogram having two of its angles at these points.

136. Shew that in every quadrilateral plane figure, two parallelograms can be described upon two opposite sides as diagonals, such that the other two diagonals shall be in the same straight line and equal.

137. Describe a quadrilateral figure whose sides shall be equal to four given straight lines. What limitation is necessary?

138. If the sides of a quadrilateral figure be bisected and the points of bisection joined, the included figure is a parallelogram, and equal in area to half the original figure.

139. A trapezium is such, that the perpendiculars let fall on a diagonal from the opposite angles are equal. Divide the trapezium into four equal triangles, by straight lines drawn to the angles from a point within it.

140. If two opposite sides of a trapezium be parallel to one another, the straight line joining their bisections, bisects the trapezium.

141. If of the four triangles into which the diagonals divide a trapezium, any two opposite ones are equal, the trapezium has two of its opposite sides parallel.

142. If two sides of a quadrilateral are parallel but not equal, and the other two sides are equal but not parallel, the opposite angles of the quadrilateral are together equal to two right angles: and conversely.

143. If two sides of a quadrilateral be parallel, and the line joining the middle points of the diagonals be produced to meet the other sides; the line so produced will be equal to half the sum of the parallel sides, and the line between the points of bisection equal to half their difference.

144. To bisect a trapezium, (1) by a line drawn from one of its angular points: (2) by a line drawn from a given point in one side.

145. To divide a square into four equal portions by lines drawn from any point in one of its sides.

146. It is impossible to divide a quadrilateral figure (except it be a parallelogram) into equal triangles by lines drawn from a point within it to its four corners.

IX.

147. If the greater of the acute angles of a right-angled triangle, be double the other, the square on the greater side is three times the square on the other.

148. Upon a given straight line construct a right-angled triangle such that the square on the other side may be equal to seven times the square on the given line.

149. If from the vertex of a plane triangle, a perpendicular fall upon the base or the base produced, the difference of the squares on the sides is equal to the difference of the squares on the segments of the base.

150. If from the middle point of one of the sides of a right-angled triangle, a perpendicular be drawn to the hypotenuse, the difference of the squares on the segments into which it is divided, is equal to the square on the other side.

151. If a straight line be drawn from one of the acute angles of a right-angled triangle, bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

152. If the sum of the squares on the three sides of a triangle be equal to eight times the square on the line drawn from the vertex to the point of bisection of the base, then the vertical angle is a right angle.

153. If a line be drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles be joined with the points where this line intersects the sides respectively opposite to them, the squares on the joining lines are together equal to the *squares on the* hypotenuse and on the line drawn parallel to it.

154. Let ACB, ADB be two right-angled triangles having a common hypotenuse AB, join CD, and on CD produced both ways draw perpendiculars AE, BF. Shew that $CE^2 + CF^2 = DE^2 + DF^2$.

155. If perpendiculars AD, BE, CF drawn from the angles on the opposite sides of a triangle intersect in G, the squares on AB, BC, and CA, are together three times the squares on AG, BG, and CG.

156. If ABC be a triangle of which the angle A is a right angle; and BE, CF be drawn bisecting the opposite sides respectively: shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

157. If ABC be an isosceles triangle, and CD be drawn perpendicular to AB; the sum of the squares on the three sides is equal to

$$AD^2 + 2.BD^2 + 3.CD^2.$$

158. The sum of the squares described upon the sides of a rhombus is equal to the squares described on its diameters.

159. A point is taken within a square, and straight lines drawn from it to the angular points of the square, and perpendicular to the sides; the squares on the first are double the sum of the squares on the last. Shew that these sums are least when the point is in the center of the square.

160. In the figure Euc. I. 47,

(*a*) Shew that the diagonals FA, AK of the squares on AB, AC, lie in the same straight line.

(*b*) If DF, EK be joined, the sum of the angles at the bases of the triangles BFD, CEK is equal to one right angle.

(*c*) If BG and CH be joined, those lines will be parallel.

(*d*) If perpendiculars be let fall from F and K on BC produced, the parts produced will be equal; and the perpendiculars together will be equal to BC.

(*e*) Join GH, KE, FD, and prove that each of the triangles so formed, equals the given triangle ABC.

(*f*) The sum of the squares on GH, KE, and FD will be equal to six times the square on the hypotenuse.

(*g*) The difference of the squares on AB, AC, is equal to the difference of the squares on AD, AE.

161. The area of any two parallelograms described on the two sides of a triangle, is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle, to the intersection of the two sides of the former parallelograms produced to meet.

162. If one angle of a triangle be a right angle, and another equal to two-thirds of a right angle, prove from the First Book of Euclid, that the equilateral triangle described on the hypotenuse, is equal to the sum of the equilateral triangles described upon the sides which contain the right angle.

BOOK II.

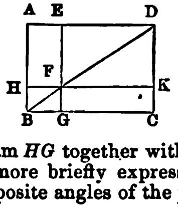
DEFINITIONS.

I.

EVERY right-angled parallelogram is called a *rectangle*, and is said to be contained by any two of the straight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter together with the two complements, is called a gnomon.



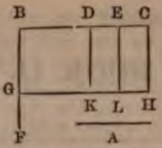
"Thus the parallelogram HG together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon."

PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines;
and let BC be divided into any parts BD, DE, EC, in the points D, E.

Then the rectangle contained by the straight lines A and BC, shall be equal to the rectangle contained by A and BD, together with that contained by A and DE, and that contained by A and EC.

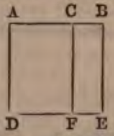


From the point B, draw BF at right angles to BC, (I. 11.)
and make BG equal to A; (I. 3.)
through G draw GH parallel to BC, (I. 31.)
and through D, E, C, draw DK, EL, CH parallel to BG, meeting GH in K, L, H.
Then the rectangle BH is equal to the rectangles BK, DL, EH.
And BH is contained by A and BC,
for it is contained by GB, BC, and GB is equal to A:
and the rectangle BK is contained by A, BD,
for it is contained by GB, BD, of which GB is equal to A:
also DL is contained by A, DE,
because DK, that is, BG, (I. 34.) is equal to A;
and in like manner the rectangle EH is contained by A, EC:
therefore the rectangle contained by A, BC, is equal to the several rectangles contained by A, BD, and by A, DE, and by A, EC.
Wherefore, if there be two straight lines, &c. Q. E. D.

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts in the point C.
Then the rectangle contained by AB, BC, together with that contained by AB, AC, shall be equal to the square on AB.

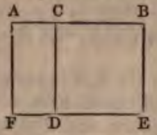


Upon AB describe the square $ADEB$, (I. 46.) and through C draw CF parallel to AD or BE, (I. 31.) meeting DE in F.
Then AE is equal to the rectangles AF, CE.
And AE is the square on AB;
and AF is the rectangle contained by BA, AC;
for it is contained by DA, AC, of which DA is equal to AB:
and CE is contained by AB, BC,
for BE is equal to AB:
therefore the rectangle contained by AB, AC, together with the *rectangle* AB, BC is equal to the square on AB.
If therefore a straight line, &c. Q.E.D.

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let the straight line *AB* be divided into any two parts in the point *C*.
Then the rectangle *AB*, *BC*, shall be equal to the rectangle *AC*, *CB*, together with the square on *BC*.

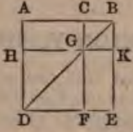


Upon *BC* describe the square *CDEB*, (I. 46.) and produce *ED* to *F*, through *A* draw *AF* parallel to *CD* or *BE*, (I. 31.) meeting *EF* in *F*.
Then the rectangle *AE* is equal to the rectangles *AD*, *CE*.
And *AE* is the rectangle contained by *AB*, *BC*,
for it is contained by *AB*, *BE*, of which *BE* is equal to *BC*:
and *AD* is contained by *AC*, *CB*, for *CD* is equal to *CB*:
and *CE* is the square on *BC*:
therefore the rectangle *AB*, *BC*, is equal to the rectangle *AC*, *CB*, together with the square on *BC*.
If therefore a straight line be divided, &c. Q.E.D.

PROPOSITION IV. THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts.

Let the straight line *AB* be divided into any two parts in *C*.
Then the square on *AB* shall be equal to the squares on *AC*, and *CB*, together with twice the rectangle contained by *AC*, *CB*.



Upon *AB* describe the square *ADEB*, (I. 46.) join *BD*,
through *C* draw *CGF* parallel to *AD* or *BE*, (I. 31.) meeting *BD* in *G* and *DE* in *F*;
and through *G* draw *HGK* parallel to *AB* or *DE*, meeting *AD* in *H*, and *BE* in *K*;
Then, because *CF* is parallel to *AD* and *BD* falls upon them,
therefore the exterior angle *BGC* is equal to the interior and opposite angle *BDA*; (I. 29.)
but the angle *BDA* is equal to the angle DBA, (I. 5.)
because *BA* is equal to *AD*, being sides of a square;

wherefore the angle *BGC* is equal to the angle *DBA* or *GBC*;
and therefore the side *BC* is equal to the side *CG*; (I. 6.)
but *BC* is equal also to *GK*, and *CG* to *BK*; (I. 34.)
wherefore the figure *CGKB* is equilateral.
It is likewise rectangular;
for, since *CG* is parallel to *BK*, and *BC* meets them,
therefore the angles *KBC*, *BCG* are equal to two right angles; (I. 29.)
but the angle *KBC* is a right angle; (def. 30. constr.)
wherefore *BCG* is a right angle:
and therefore also the angles *CGK*, *GKB*, opposite to these, are right angles; (I. 34.)
wherefore *CGKB* is rectangular:
but it is also equilateral, as was demonstrated;
wherefore it is a square, and it is upon the side *CB*.
For the same reason *HF* is a square,
and it is upon the side *HG*, which is equal to *AC*. (I. 34.)
Therefore the figures *HF*, *CK*, are the squares on *AC*, *CB*.
And because the complement *AG* is equal to the complement *GE*, (I. 43.)
and that *AG* is the rectangle contained by *AC*, *CB*,
for *GC* is equal to *CB*;
therefore *GE* is also equal to the rectangle *AC*, *CB*;
wherefore *AG*, *GE* are equal to twice the rectangle *AC*, *CB*;
and *HF*, *CK* are the squares on *AC*, *CB*;
wherefore the four figures *HF*, *CK*, *AG*, *GE*, are equal to the squares on *AC*, *CB*, and twice the rectangle *AC*, *CB*:
but *HF*, *CK*, *AG*, *GE* make up the whole figure *ADEB*, which is the square on *AB*;
therefore the square on *AB* is equal to the squares on *AC*, *CB*, and twice the rectangle *AC*, *CB*.
Wherefore, if a straight line be divided, &c. Q.E.D.

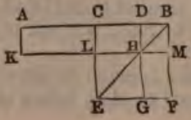
COR. From the demonstration, it is manifest, that the parallelograms about the diameter of a square, are likewise squares.

PROPOSITION V. THEOREM.

If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line *AB* be divided into two equal parts in the point *C*, and into two unequal parts in the point *D*.

Then the rectangle *AD*, *DB*, together with the square on *CD*, shall be equal to the square on *CB*.



Upon *CB* describe the square *CEFB*, (I. 46.) join *BE*,

through *D* draw *DHG* parallel to *CE* or *BF*, (I. 31.) meeting *DE* in *H*, and *EF* in *G*,

and through *H* draw *KLM* parallel to *CB* or *EF*, meeting *CE* in *L*, and *BF* in *M*;

also through *A* draw *AK* parallel to *CL* or *BM*, meeting *MLK* in *K*.

Then because the complement *CH* is equal to the complement *HF*, (I. 43.) to each of these equals add *DM*;

therefore the whole *CM* is equal to the whole *DF*;

but because the line *AC* is equal to *CB*,

therefore *AL* is equal to *CM*, (I. 36.)

therefore also *AL* is equal to *DF*;

to each of these equals add *CH*,

and therefore the whole *AH* is equal to *DF* and *CH*:

but *AH* is the rectangle contained by *AD*, *DB*, for *DH* is equal to *DB*;

and *DF* together with *CH* is the gnomon *CMG*;

therefore the gnomon *CMG* is equal to the rectangle *AD*, *DB*:

to each of these equals add *LG*, which is equal to the square on *CD*; (II. 4. Cor.)

therefore the gnomon *CMG*, together with *LG*, is equal to the rectangle *AD*, *DB*, together with the square on *CD*:

but the gnomon *CMG* and *LG* make up the whole figure *CEFB*, which is the square on *CB*;

therefore the rectangle *AD*, *DB*, together with the square on *CD* is equal to the square on *CB*.

Wherefore, if a straight line, &c. Q. E. D.

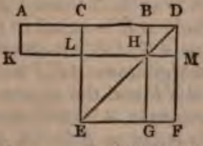
COR. From this proposition it is manifest, that the difference of the squares on two unequal lines *AC*, *CD*, is equal to the rectangle contained by their sum *AD* and their difference *DB*.

PROPOSITION VI. THEOREM.

If a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line *AB* be bisected in *C*, and produced to the point *D*.

Then the rectangle *AD*, *DB*, together with the square on *CB*, shall be equal to the square on *CD*.



Upon *CD* describe the square *CEFD*, (I. 46.) and join *DE*,

through *B* draw *BHG* parallel to *CE* or *DF*, (I. 31.) meeting *DE* in *H*, and *EF* in *G*;

through *H* draw *KLM* parallel to AD or EF, meeting DF in *M*, and *CE* in *L*;

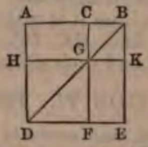
and through A draw *AK* parallel to CL or DM, meeting MLK in K.

Then because the line AC is equal to CB,
therefore the rectangle AL is equal to the rectangle CH, (I. 36.)
but CH is equal to HF; (I. 43.)
therefore AL is equal to HF;
to each of these equals add CM;
therefore the whole AM is equal to the gnomon CMG:
but AM is the rectangle contained by AD, DB,
for DM is equal to DB: (II. 4. Cor.)
therefore the gnomon CMG is equal to the rectangle AD, DB:
to each of these equals add LG which is equal to the square on CB;
therefore the rectangle AD, DB, together with the square on CB, is equal to the gnomon CMG, and the figure LG;
but the gnomon CMG and LG make up the whole figure $CEFD$, which is the square on CD;
therefore the rectangle AD, DB, together with the square on CB, is equal to the square on CD.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION VII. THEOREM.

If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts in the point C.
Then the squares on AB, BC shall be equal to twice the rectangle AB, BC, together with the square on AC.



Upon AB describe the square $ADEB$, (I. 46.) and join BD;
through C draw CF parallel to AD or BE (I. 31.) meeting BD in G, and DE in F;
through G draw HGK parallel to AB or DE, meeting AD in H, and BE in K.
Then because AG is equal to GE, (I. 43.)
add to each of them CK;
therefore the whole AK is equal to the whole CE;
and therefore AK, CE, are double of AK:
but AK, CE, are the gnomon AKF and the square CK;
therefore the gnomon AKF and the square CK are double of AK:
but twice the rectangle AB, BC, is double of AK,
for BK is equal to BC; (II. 4. Cor.)
therefore the gnomon AKF and the square CK, are equal to twice the rectangle AB, BC;
to each of these equals add HF, which is equal to the square on AC,
therefore the gnomon AKF, and the squares CK, HF, are equal to twice the rectangle AB, BC, and the square on AC;
but the gnomon AKF, together with the squares CK, HF, make

up the whole figure $ADEB$ and CK, which are the squares on AB and BC;

therefore the squares on AB and BC are equal to twice the rectangle AB, BC, together with the square on AC.

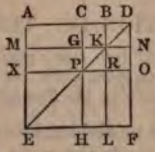
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square on the other part, is equal to the square on the straight line, which is made up of the whole and that part.

Let the straight line AB be divided into any two parts in the point C.

Then four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.



Produce AB to D, so that BD be equal to CB, (I. 3.)
upon AD describe the square $AEFD$, (I. 46.) and join DE,
through B, C, draw BL, CH parallel to AE or DF, and cutting DE in the points K, P respectively, and meeting EF in L, H;
through K, P, draw $MGKN$, $XPRO$ parallel to AD or EF.
Then because CB is equal to BD, CB to GK, and BD to KN;
therefore GK is equal to KN;
for the same reason, PR is equal to RO;
and because CB is equal to BD, and GK to KN,
therefore the rectangle CK is equal to BN, and GR to RN; (I. 36.)
but CK is equal to RN, (I. 43.)
because they are the complements of the parallelogram CO;
therefore also BN is equal to GR;
and the four rectangles BN, CK, GR, RN, are equal to one another, and so are quadruple of one of them CK.

Again, because CB is equal to BD, and BD to BK, that is, to CG;
and because CB is equal to GK, that is, to GP;
therefore CG is equal to GP.

And because CG is equal to GP, and PR to RO,
therefore the rectangle AG is equal to MP, and PL to RF;
but the rectangle MP is equal to PL, (I. 43.)
because they are the complements of the parallelogram ML:
wherefore also AG is equal to RF:
therefore the four rectangles AG, MP, PL, RF, are equal to one another, and so are quadruple of one of them AG.

And it was demonstrated, that the four CK, BN, GR, and RN, are quadruple of CK:

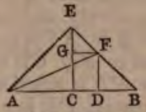
therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK.
And because AK is the rectangle contained by AB, BC,
for BK is equal to BC;
therefore four times the rectangle AB, BC is quadruple of AK:
but the gnomon AOH was demonstrated to be quadruple of AK;
therefore four times the rectangle AB, BC is equal to the gnomon AOH;
to each of these equals add XH, which is equal to the square on AC;
therefore four times the rectangle AB, BC, together with the square on AC, is equal to the gnomon AOH and the square XH;
but the gnomon AOH and XH make up the figure $AEFD$, which is the square on AD;
therefore four times the rectangle AB, BC together with the square on AC, is equal to the square on AD, that is, on AB and BC added together in one straight line.
Wherefore, if a straight line, &c, Q.E.D.

PROPOSITION IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts; the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D.

Then the squares on AD, DB together, shall be double of the squares on AC, CD.



From the point C draw CE at right angles to AB, (I. 11.)
make CE equal to AC or CB, (I. 3.) and join EA, EB;
through D draw DF parallel to CE, meeting EB in F, (I. 31.)
through F draw FG parallel to BA, and join AF.
Then, because AC is equal to CE,
therefore the angle AEC is equal to the angle EAC; (I. 5.)
and because ACE is a right angle,
therefore the two other angles AEC, EAC of the triangle are together equal to a right angle; (I. 32.)
and since they are equal to one another;
therefore each of them is half a right angle.
For the same reason, each of the angles CEB, EBC is half a right angle;
and therefore the whole AEB is a right angle.
And because the angle GEF is half a right angle,
and EGF a right angle,
for it is equal to the interior and opposite angle ECB, (I. 29.)
therefore the remaining angle EFG is half a right angle;
wherefore the angle GEF is equal to the angle EFG,
and the side GF equal to the side EG. (I. 6.)

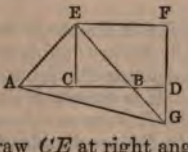
Again, because the angle at B is half a right angle,
and FDB a right angle,
for it is equal to the interior and opposite angle ECB, (I. 29.)
therefore the remaining angle BFD is half a right angle;
wherefore the angle at B is equal to the angle BFD,
and the side DF equal to the side DB. (I. 6.)
And because AC is equal to CE,
the square on AC is equal to the square on CE;
therefore the squares on AC, CE are double of the square on AC;
but the square on AE is equal to the squares on AC, CE, (I. 47.)
because ACE is a right angle;
therefore the square on AE is double of the square on AC.
Again, because EG is equal to GF,
the square on EG is equal to the square on GF;
therefore the squares on EG, GF are double of the square on GF;
but the square on EF is equal to the squares on EG, GF; (I. 47.)
therefore the square on EF is double of the square on GF;
and GF is equal to CD; (I. 34.)
therefore the square on EF is double of the square on CD;
but the square on AE is double of the square on AC;
therefore the squares on AE, EF are double of the squares on AC, CD;
but the square on AF is equal to the squares on AE, EF,
because AEF is a right angle: (I. 47.)
therefore the square on AF is double of the squares on AC, CD:
but the squares on AD, DF are equal to the square on AF;
because the angle ADF is a right angle; (I. 47.)
therefore the squares on AD, DF are double of the squares on AC, CD;
and DF is equal to DB;
therefore the squares on AD, DB are double of the squares on AC, CD.
If therefore a straight line be divided, &c. Q.E.D.

PROPOSITION X. THEOREM.

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D.

Then the squares on AD, DB, shall be double of the squares on AC, CD.



From the point C draw CE at right angles to AB, (I. 11.)
make CE equal to AC or CB, (I. 3.) and join AE, EB;
through E draw EF parallel to AB, (I. 31.)
and through D draw DF parallel to CE, meeting EF in F.

Then because the straight line *EF* meets the parallels *CE*, *FD*, therefore the angles *CEF*, *EFD* are equal to two right angles; (I. 29.) and therefore the angles *BEF*, *EFD* are less than two right angles.

But straight lines, which with another straight line make the interior angles upon the same side of a line, less than two right angles, will meet if produced far enough; (I. ax. 12.)

therefore *EB*, *FD* will meet, if produced towards *B*, *D*;
let them be produced and meet in *G*, and join *AG*.
Then, because *AC* is equal to *CE*,
therefore the angle *CEA* is equal to the angle *EAC*; (I. 5.)
and the angle *ACE* is a right angle;
therefore each of the angles *CEA*, *EAC* is half a right angle. (I. 32.)

For the same reason,
each of the angles *CEB*, *EBC* is half a right angle;
therefore the whole *AEB* is a right angle.
And because *EBC* is half a right angle,
therefore *DBG* is also half a right angle, (I. 15.)
for they are vertically opposite;
but *BDG* is a right angle,
because it is equal to the alternate angle *DCE*; (I. 29.)
therefore the remaining angle *DGB* is half a right angle;
and is therefore equal to the angle *DBG*;
wherefore also the side *BD* is equal to the side *DG*. (I. 6.)

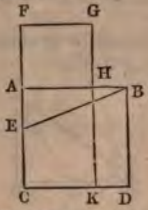
Again, because *EGF* is half a right angle, and the angle at *F* is a right angle, being equal to the opposite angle *ECD*, (I. 34.)
therefore the remaining angle *FEG* is half a right angle,
and therefore equal to the angle *EGF*;
wherefore also the side *GF* is equal to the side *FE*. (I. 6.)
And because *EC* is equal to *CA*;
the square on *EC* is equal to the square on *CA*;
therefore the squares on *EC*, *CA* are double of the square on *CA*;
but the square on *EA* is equal to the squares on *EC*, *CA*; (I. 47.)
therefore the square on *EA* is double of the square on *AC*.
Again, because *GF* is equal to *FE*,
the square on *GF* is equal to the square on *FE*;
therefore the squares on *GF*, *FE* are double of the square on *FE*;
but the square on *EG* is equal to the squares on *GF*, *FE*; (I. 47.)
therefore the square on *EG* is double of the square on *FE*;
and *FE* is equal to *CD*; (I. 34.)
wherefore the square on *EG* is double of the square on *CD*;
but it was demonstrated,
that the square on *EA* is double of the square on *AC*;
therefore the squares on *EA*, *EG* are double of the squares on *AC*, *CD*;
but the square on *AG* is equal to the squares on *EA*, *EG*; (I. 47.)
therefore the square on *AG* is double of the squares on *AC*, *CD*:
but the squares on *AD*, *DG* are equal to the square on *AG*;
therefore the squares on *AD*, *DG* are double of the squares on *AC*, *CD*;
but *DG* is equal to *DB*;
therefore the squares on AD, DB are double of the squares on *AC*, *CD*.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XI. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square on the other part.

Let AB be the given straight line.

It is required to divide AB into two parts, so that the rectangle contained by the whole line and one of the parts, shall be equal to the square on the other part.



Upon AB describe the square $ACDB$; (I. 46.)
bisect AC in E, (I. 10.) and join BE,
produce CA to F, and make EF equal to EB, (I. 3.)
upon AF describe the square $FGHA$. (I. 46.)

Then AB shall be divided in H, so that the rectangle AB, BH is equal to the square on AH.

Produce GH to meet CD in K.

Then because the straight line AC is bisected in E, and produced to F,
therefore the rectangle CF, FA together with the square on AE, is equal to the square on EF; (II. 6.)
but EF is equal to EB;
therefore the rectangle CF, FA together with the square on AE, is equal to the square on EB;
but the squares on BA, AE are equal to the square on EB, (I. 47.)
because the angle EAB is a right angle;
therefore the rectangle CF, FA, together with the square on AE, is equal to the squares on BA, AE;
take away the square on AE, which is common to both;
therefore the rectangle contained by CF, FA is equal to the square on BA.

But the figure FK is the rectangle contained by CF, FA,
for FA is equal to FG;
and AD is the square on AB;
therefore the figure FK is equal to AD;
take away the common part AK,
therefore the remainder FH is equal to the remainder HD;
but HD is the rectangle contained by AB, BH,
for AB is equal to BD;
and FH is the square on AH;
therefore the rectangle AB, BH, is equal to the square on AH.

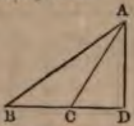
Wherefore the straight line AB is divided in H, so that the *rectangle* AB, BH is equal to the square on AH. Q.E.F.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle, is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A, let AD be drawn perpendicular to BC produced.

Then the square on AB shall be greater than the squares on AC, CB, by twice the rectangle BC, CD.



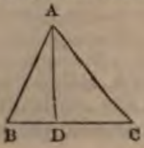
Because the straight line BD is divided into two parts in the point C,
therefore the square on BD is equal to the squares on BC, CD, and twice the rectangle BC, CD; (II. 4.)
to each of these equals add the square on DA;
therefore the squares on BD, DA are equal to the squares on BC, CD, DA, and twice the rectangle BC, CD;
but the square on BA is equal to the squares on BD, DA, (I. 47.)
because the angle at D is a right angle;
and the square on CA is equal to the squares on CD, DA;
therefore the square on BA is equal to the squares on BC, CA, and twice the rectangle BC, CD;
that is, the square on BA is greater than the squares on BC, CA, by twice the rectangle BC, CD.
Therefore in obtuse-angled triangles, &c. Q.E.D.

PROPOSITION XIII. THEOREM.

In every triangle, the square on the side subtending either of the acute angles, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle. (I. 12.)

Then the square on AC opposite to the angle B, shall be less than the squares on CB, BA, by twice the rectangle CB, BD.



First, let AD fall within the triangle ABC.

Then because the straight line CB is divided into two parts in D, the squares on CB, BD are equal to twice the rectangle contained by CB, BD, and the square on DC; (II. 7.)

to each of these equals add the square on AD;

therefore the squares on CB, BD, DA, are equal to twice the rectangle CB, BD, and the squares on AD, DC;

but the square on AB is equal to the squares on BD, DA, (I. 47.)

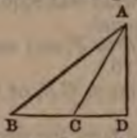
because the angle BDA is a right angle;

and the square on AC is equal to the squares on AD, DC;

therefore the squares on CB, BA are equal to the square on AC, and twice the rectangle CB, BD:

that is, the square on AC alone is less than the squares on CB, BA, by twice the rectangle CB, BD.

Secondly, let AD fall without the triangle ABC.



Then, because the angle at D is a right angle,

the angle ACB is greater than a right angle; (I. 16.)

and therefore the square on AB is equal to the squares on AC, CB, and twice the rectangle BC, CD; (II. 12.)

to each of these equals add the square on BC;

therefore the squares on AB, BC are equal to the square on AC, twice the square on BC, and twice the rectangle BC, CD;

but because BD is divided into two parts in C,

therefore the rectangle DB, BC is equal to the rectangle BC, CD, and the square on BC; (II. 3.)

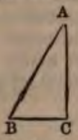
and the doubles of these are equal;

that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square on BC:

therefore the squares on AB, BC are equal to the square on AC, and twice the rectangle DB, BC:

wherefore the square on AC alone is less than the squares on AB, BC; by twice the rectangle DB, BC.

Lastly, let the side AC be perpendicular to BC.



Then BC is the straight line between the perpendicular and the acute angle at B;

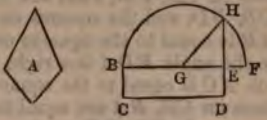
and it is manifest, that the squares on AB, BC, are equal to the square on AC, and twice the square on BC. (I. 47.)

Therefore in any triangle, &c. Q.E.D.

PROPOSITION XIV. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let *A* be the given rectilineal figure.
It is required to describe a square that shall be equal to *A*.



Describe the rectangular parallelogram *BCDE* equal to the rectilineal figure *A*. (I. 45.)
Then, if the sides of it, *BE*, *ED*, are equal to one another,
it is a square, and what was required is now done.
But if *BE*, *ED*, are not equal,
produce one of them *BE* to *F*, and make *EF* equal to *ED*,
bisect *BF* in *G*; (I. 10.)
from the center *G*, at the distance *GB*, or *GF*, describe the semicircle *BHF*,
and produce *DE* to meet the circumference in *H*.
The square described upon *EH* shall be equal to the given rectilineal figure *A*.

Join *GH*.

Then because the straight line *BF* is divided into two equal parts in the point *G*, and into two unequal parts in the point *E*;
therefore the rectangle *BE*, *EF*, together with the square on *EG*, is equal to the square on *GF*; (II. 5.)
but *GF* is equal to *GH*; (def. 15.)
therefore the rectangle *BE*, *EF*, together with the square on *EG*, is equal to the square on *GH*;
but the squares on *HE*, *EG* are equal to the square on *GH*; (I. 47.)
therefore the rectangle *BE*, *EF*, together with the square on *EG*, is equal to the squares on *HE*, *EG*;
take away the square on *EG*, which is common to both;
therefore the rectangle *BE*, *EF* is equal to the square on *HE*.
But the rectangle contained by *BE*, *EF* is the parallelogram *BD*, because *EF* is equal to *ED*;
therefore *BD* is equal to the square on *EH*;
but *BD* is equal to the rectilineal figure *A*; (constr.)
therefore the square on *EH* is equal to the rectilineal figure *A*.
Wherefore a square has been made equal to the given rectilineal figure *A*, namely, the square described upon *EH*. Q.E.F.

NOTES TO BOOK II.

In Book I, Geometrical magnitudes of the same kind, lines, angles and surfaces, more particularly triangles and parallelograms, are compared, either as being absolutely equal, or unequal to one another.

In Book II, the properties of right-angled parallelograms, but without reference to their magnitudes, are demonstrated, and an important extension is made of Euc. I. 47, to acute-angled and obtuse-angled triangles. Euclid has given no definition of *a rectangular parallelogram* or *rectangle:* probably, because the Greek expression *παραλληλόγραμμον ὀρθογώνιον*, or *ὀρθογώνιον* simply, is a definition of the figure. In English, the term *rectangle*, formed from *rectus angulus*, ought to be defined before its properties are demonstrated. A rectangle may be defined to be a parallelogram having one angle a right-angle, or a right angled parallelogram; and a square is a rectangle having all its sides equal.

As the squares in Euclid's demonstrations are squares described or supposed to be described on straight lines, the expression "*the square on AB*," is a more appropriate abbreviation for "*the square described on the line AB*," than "*the square of AB*." The latter expression more fitly expresses the arithmetical or algebraical equivalent for the square on the line *AB*.

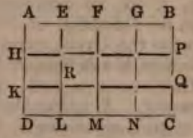
In Euc. I. 35, it may be seen that there may be an indefinite number of parallelograms on the same base and between the same parallels whose areas are always equal to one another; but that one of them has all its angles right angles, and the length of its boundary less than the boundary of any other parallelogram upon the same base and between the same parallels. The area of this rectangular parallelogram is therefore determined by the two lines which contain one of its right angles. Hence it is stated in Def. 1, that every right-angled parallelogram *is said to be contained* by any two of the straight lines which contain one of the right angles. No distinction is made in Book II, between *equality* and *identity*, as the rectangle may be said to be contained by two lines which are equal respectively to the two which contain one right angle of the figure. It may be remarked that the rectangle itself *is bounded* by four straight lines.

It is of primary importance to discriminate the Geometrical conception of a rectangle from the Arithmetical or Algebraical representation of it. The subject of Geometry is *magnitude* not *number*, and therefore it would be a departure from strict reasoning on space, to substitute in Geometrical demonstrations, the Arithmetical or Algebraical representation of a rectangle for the rectangle itself. It is however, absolutely necessary that the connexion of *number* and *magnitude* be clearly understood, as far as regards the representation of lines and areas.

All lines are measured by lines, and all surfaces by surfaces. Some one line of definite length is arbitrarily assumed as the linear unit, and the length of every other line is represented by the number of linear units, contained in it. The square is the figure assumed for the measure of surfaces. The square unit or the unit of area is assumed to be that square, the side of which is one unit in length, and the magnitude of every surface is represented by the number of square units contained in it. But here it may be remarked, that the properties of rectangles and *squares in the* Second Book of Euclid are proved independently of the consideration, whether the sides of the rectangles can be represented by any multiples of the same linear unit. If, however, the sides of rectangles are supposed to be divisible into an exact number of linear units, a numerical representation for the area of a rectangle may be deduced.

On two lines at right angles to each other, take AB equal to 4, and AD equal to 3 linear units.

Complete the rectangle $ABCD$, and through the points of division of AB, AD, draw EL, FM, GN parallel to AD; and HP, KQ parallel to AB respectively.



Then the whole rectangle AC is divided into squares, all equal to each other.

And AC is equal to the sum of the rectangles AL, EM, FN, GC; (II. 1.)
also these rectangles are equal to one another, (I. 36.)
therefore the whole AC is equal to four times one of them AL.

Again, the rectangle AL is equal to the rectangles EH, HR, RD, and these rectangles, by construction, are squares described upon the equal lines AH, HK, KD, and are equal to one another.

Therefore the rectangle AL is equal to 3 times the square on AH,
but the whole rectangle AC is equal to 4 times the rectangle AL,
therefore the rectangle AC is 4×3 times the square on AH, or 12 square units:

that is, the product of the two numbers which express the number of linear units in the two sides, will give the number of square units in the rectangle, and therefore will be an arithmetical representation of its area.

And generally, if AB, AD, instead of 4 and 3, consisted of a and b linear units respectively, it may be shewn in a similar manner, that the area of the rectangle AC would contain ab square units; and therefore the product ab is a proper representation for the area of the rectangle AC.

Hence, it follows, that the term *rectangle* in Geometry corresponds to the term *product* in Arithmetic and Algebra, and that a similar comparison may be made between the products of the two numbers which represent the sides of rectangles, as between the areas of the rectangles themselves. This forms the basis of what are called Arithmetical or Algebraical proofs of Geometrical properties.

If the two sides of the rectangle be equal, or if b be equal to a, the figure is a square, and the area is represented by aa or a^2.

Also, since a triangle is equal to the half of a parallelogram of the same base and altitude;

Therefore the area of a triangle will be represented by half the rectangle which has the same base and altitude as the triangle: in other words, if the length of the base be a units, and the altitude be b units;

Then the area of the triangle is algebraically represented by $\frac{1}{2}ab$.

The demonstrations of the first eight propositions, exemplify the obvious axiom, that, "the whole area of every figure in each case, is *equal to all the parts* of it taken together."

Def. 2. The parallelogram EK together with the complements AF

FC, is also a *gnomon*, as well as the parallelogram *HG* together with the same complements.

Prop. I. For the sake of brevity of expression, "the rectangle contained by the straight lines *AB*, *BC*," is called "the rectangle *AB*, *BC*;" and sometimes "the rectangle *ABC*."

To this proposition may be added the corollary: If two straight lines be divided into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the several parts of one line and the several parts of the other respectively.

The method of reasoning on the properties of rectangles, by means of the products which indicate the number of square units contained in their areas, is foreign to Euclid's ideas of rectangles, as discussed in his Second Book, which have no reference to any particular unit of length or measure of surface.

Prop. I. The figures *BH*, *BK*, *DL*, *EH* are rectangles, as may readily be shewn. For, by the parallels, the angle *CEL* is equal to *EDK*; and the angle *EDK* is equal to *BDG* (Euc. I. 29.). But *BDG* is a right angle. Hence one of the angles in each of the figures *BH*, *BK*, *DL*, *EH* is a right angle, and therefore (Euc. I. 46, Cor.) these figures are rectangular.

Prop. I. Algebraically. (fig. Prop. I.)

Let the line *BC* contain a linear units, and the line *A*, b linear units of the same length.

Also suppose the parts *BD*, *DE*, *EC* to contain m, n, p linear units respectively.

$$\text{Then } a = m + n + p,$$

multiply these equals by b,

$$\text{therefore } ab = bm + bn + bp.$$

That is, the product of two numbers, one of which is divided into any number of parts, is equal to the sum of the products of the undivided number, and the several parts of the other;

or, if the Geometrical interpretation of the products be restored,

The number of square units expressed by the product ab, is equal to the number of square units expressed by the sum of the products bm, bn, bp.

Prop. II. Algebraically. (fig. Prop. II.)

Let *AB* contain a linear units, and *AC*, *CB*, m and n linear units respectively.

$$\text{Then } m + n = a,$$

multiply these equals by a,

$$\text{therefore } am + an = a^2.$$

That is, if a number be divided into any two parts, the sum of the products of the whole and each of the parts is equal to the square of the whole number

Prop. III. Algebraically. (fig. Prop. III.)

Let *AB* contain a linear units, and let *BC* contain m, and *AC*, n linear units.

$$\text{Then } a = m + n,$$

multiply these equals by m,

$$\text{therefore } ma = m^2 + mn.$$

That is, if a number be divided into any two parts, the product of the whole number and one of the parts, is equal to the square of that part, and the product of the two parts.

Prop. iv. might have been deduced from the two preceding propositions; but Euclid has preferred the method of exhibiting, in the demonstrations of the second book, the equality of the spaces compared.

In the corollary to Prop. xlvi. Book I, it is stated that a parallelogram which has one right angle, has all its angles right angles. By applying this corollary, the demonstration of Prop. iv. may be considerably shortened.

If the two parts of the line be equal, then the square on the whole line is equal to four times the square on half the line.

Also, if a line be divided into any three parts, the square on the whole line is equal to the squares on the three parts, and twice the rectangles contained by every two parts.

Prop. iv. Algebraically (fig. Prop. iv.)

Let the line AB contain a linear units, and the parts of it AC and BC, m and n linear units respectively.

$$\text{Then } a = m + n,$$

$$\text{squaring these equals, } \therefore a^2 = (m + n)^2,$$

$$\text{or } a^2 = m^2 + 2mn + n^2.$$

That is, if a number be divided into any two parts, the square of the number is equal to the squares of the two parts together with twice the product of the two parts.

From Euc. ii. 4, may be deduced a proof of Euc. i. 47. In the fig. take DL on DE, and EM on EB, each equal to BC, and join CH, HL, LM, MC. Then the figure $HLMC$ is a square, and the four triangles CAH, HDL, LEM, MBC are equal to one another, and together are equal to the two rectangles AG, GE.

Now AG, GE, FH, CK are together equal to the whole figure $ADEB$; and $HLMC$, with the four triangles CAH, HDL, LEB, MBC also make up the whole figure $ADEB$;

Hence AG, GE, FH, CK are equal to $HLMC$ together with the four triangles:

but AG, GE are equal to the four triangles.

wherefore FH, CK are equal to $HLMC$,

that is, the squares on AC, AH are together equal to the square on CH.

Prop. v. It must be kept in mind, that the sum of two straight lines in Geometry, means the straight line formed by joining the two lines together, so that both may be in the same straight line.

The following simple properties respecting the equal and unequal division of a line are worthy of being remembered.

I. Since $AB = 2BC = 2(BD + DC) = 2BD + 2DC$. (fig. Prop. v.)

$$\text{and } AB = AD + DB;$$

$$\therefore 2CD + 2DB = AD + DB,$$

and by subtracting $2DB$ from these equals,

$$\therefore 2CD = AD - DB,$$

$$\text{and } CD = \tfrac{1}{2}(AD - DB).$$

That is, if a line AB be divided into two equal parts in C, and into two unequal parts in D, the part CD of the line between the points of section is equal to half the difference of the unequal parts AD and DB.

II. *Here* $AD = AC + CD$, the sum of the unequal parts, (fig. Prop. v.)

and $DB = AC - CD$ their difference.

Hence by adding these equals together,

$$\therefore AD + DB = 2AC,$$

or the sum and difference of two lines AC, CD, are together equal to twice the greater line.

And the halves of these equals are equal,

$$\therefore \tfrac{1}{2}.AD + \tfrac{1}{2}.DB = AC,$$

or, half the sum of two unequal lines AC, CD added to half their difference is equal to the greater line AC.

III. Again, since $AD = AC + CD$, and $DB = AC - CD$, by subtracting these equals,

$$\therefore AD - DB = 2CD,$$

or, the difference between the sum and difference of two unequal lines is equal to twice the less line.

And the halves of these equals are equal,

$$\therefore \tfrac{1}{2}.AD - \tfrac{1}{2}.DB = CD,$$

or, half the difference of two lines subtracted from half their sum is equal to the less of the two lines.

IV. Since $AC - CD = DB$ the difference,

$$\therefore AC = CD + DB,$$

and adding CD the less to each of these equals,

$$\therefore AC + CD = 2CD + DB,$$

or, the sum of two unequal lines is equal to twice the less line together with the difference between the lines.

Prop. v. Algebraically.

Let AB contain $2a$ linear units,

its half BC will contain a linear units.

And let CD the line between the points of section contain m linear units. Then AD the greater of the two unequal parts, contains $a+m$ linear units; and DB the less contains $a-m$ units.

Also m is half the difference of $a+m$ and $a-m$;

$$\therefore (a+m)(a-m) = a^2 - m^2,$$

to each of these equals add m^2;

$$\therefore (a+m)(a-m) + m^2 = a^2.$$

That is, if a number be divided into two equal parts, and also into two unequal parts, the product of the unequal parts together with the square of half their difference, is equal to the square of half the number.

Bearing in mind that AC, CD are respectively half the sum and half the difference of the two lines AD, DB; the corollary to this proposition may be expressed in the following form: "The rectangle contained by two straight lines is equal to the difference on the squares of half their sum and half their difference."

The rectangle contained by AD and DB, and the square on BC are each bounded by the same extent of line, but the spaces enclosed differ by the square on CD.

A given straight line is said *to be produced* when it has its length increased in either direction, and the increase it receives, is called the *part produced*.

If a point be taken in a line or in a line produced, the line is said to be divided *internally* or *externally*, and the distances of the point from the ends of the line are called the internal or external segments of the line, according as the point of section is in the line or the line produced.

Prop. VI. Algebraically.

Let AB contain $2a$ linear units, then its half BC contains a units; and let BD contain m units.

$$\text{Then } AD \text{ contains } 2a + m \text{ units,}$$
$$\text{and } \therefore (2a + m)\,m = 2am + m^2;$$
$$\text{to each of these equals add } a^2,$$
$$\therefore (2a + m)\,m + a^2 = a^2 + 2am + m^2.$$
$$\text{But } a^2 + 2am + n^2 = (a + m)^2,$$
$$\therefore (2a + m)\,m + a^2 = (a + m)^2.$$

That is, If a number be divided into two equal numbers, and another number be added to the whole and to one of the parts; the product of the whole number thus increased and the other number, together with the square of half the given number, is equal to the square of the number which is made up of half the given number increased.

The algebraical results of Prop. V. and Prop. VI. are identical, as it is obvious that the difference of $a + m$ and $a - m$ in Prop. V. is equal to the difference of $2a + m$ and m in Prop. VI, and one algebraical result expresses the truth of both propositions.

This arises from the two ways in which the difference between two unequal lines may be represented geometrically, when they are in the same direction.

In the diagram (fig. to Prop. V.), the difference DB of the two unequal lines AC and CD is exhibited by producing the less line CD, and making CB equal to AC the greater.

Then the part produced DB is the difference between AC and CD,
for AC is equal to CB, and taking CD from each,
the difference of AC and CD is equal to the difference of CB and CD.

In the diagram (fig. to Prop. VI.), the difference DB of the two unequal lines CD and CA is exhibited by cutting off from CD the greater, a part CB equal to CA the less.

Prop. VII. Either of the two parts AC, CB of the line AB may be taken: and it is equally true, that the squares on AB and AC are equal to twice the rectangle AB, AC, together with the square on BC.

Prop. VII. Algebraically.

Let AB contain a linear units, and let the parts AC and CB contain m and n linear units respectively.

$$\text{Then } a = m + n;$$
$$\text{squaring these equals,}$$
$$\therefore a^2 = m^2 + 2mn + n^2,$$
$$\text{add } n^2 \text{ to each of these equals,}$$
$$\therefore a^2 + n^2 = m^2 + 2mn + 2n^2.$$
$$\text{But } 2mn + 2n^2 = 2\,(m + n)\,n = 2an,$$
$$\therefore a^2 + n^2 = m^2 + 2an.$$

That is, If a number be divided into any two parts, the squares of the whole number and of one of the parts are equal to twice the product of the whole number and that part, together with the square of the other part.

Prop. VIII. As in Prop. VII. either part of the line may be taken, *and it is also true in this* Proposition, that four times the rectangle contained by *AB*, *AC* together with the square on *BC*, is equal to the square on the straight line made up of *AB* and *AC* together.

The truth of this proposition may be deduced from Euc. II. 4 and 7.

For the square on *AD* (fig. Prop. 8.) is equal to the squares on *AB*, *BD*, and twice the rectangle *AB*, *BD*; (Euc. II. 4.) or the squares on *AB*, *BC*, and twice the rectangle *AB*, *BC*, because *BC* is equal to *BD*: and the squares on *AB*, *BC* are equal to twice the rectangle *AB*, *BC* with the square on *AC*: (Euc. II. 7.) therefore the square on *AD* is equal to four times the rectangle *AB*, *BC* together with the square on *AC*.

Prop. VIII. Algebraically.

Let the whole line *AB* contain a linear units of which the parts *AC*, *CB* contain m, n units respectively.

$$\text{Then } m + n = a,$$

and subtracting or taking n from each,

$$\therefore m = a - n,$$

squaring these equals,

$$\therefore m^2 = a^2 - 2an + n^2,$$

and adding $4an$ to each of these equals,

$$\therefore 4an + m^2 = a^2 + 2an + n^2.$$

$$\text{But } a^2 + 2an + n^2 = (a + n)^2,$$

$$\therefore 4an + m^2 = (a + n)^2.$$

That is, If a number be divided into any two parts, four times the product of the whole number and one of the parts, together with the square of the other part, is equal to the square of the number made of the whole and the part first taken.

Prop. VIII. may be put under the following form: The square on the sum of two lines exceeds the square on their difference, by four times the rectangle contained by the lines.

Prop. IX. The demonstration of this proposition may be deduced from Euc. II. 4 and 7.

For (Euc. II. 4.) the square on *AD* is equal to the squares on *AC*, *CD* and twice the rectangle *AC*, *CD*; (fig. Prop. 9.) and adding the square on *DB* to each, therefore the squares on *AD*, *DB* are equal to the squares on *AC*, *CD* and twice the rectangle *AC*, *CD* together with the square on *DB*; or to the squares on *BC*, *CD* and twice the rectangle *BC*, *CD* with the square on *DB*, because *BC* is equal to *AC*.

But the squares on *BC*, *CD* are equal to twice the rectangle *BC*, *CD*, with the square on *DB*. (Euc. II. 7.)

Wherefore the squares on *AD*, *DB* are equal to twice the squares on *BC* and *CD*.

Prop. IX. Algebraically.

Let *AB* contain $2a$ linear units, its half *AC* or *BC* will contain a units; and let *CD* the line between the points of section contain m units.

Also *AD* the greater of the two unequal parts contains $a + m$ units,

and *DB* the less contains $a - m$ units.

$$\text{Then } (a + m)^2 = a^2 + 2am + m^2,$$

$$\text{and } (a - m)^2 = a^2 - 2am + m^2.$$

Hence by adding these equals,

$$\therefore (a + m)^2 + (a - m)^2 = 2a^2 + 2m^2.$$

That is, If a number be divided into two equal parts, and also into two unequal parts, the sum of the squares of the two unequal parts is equal to twice the square of half the number itself, and twice the square of half the difference of the unequal parts.

The proof of Prop. x. may be deduced from Euc. II. 4, 7, as Prop. IX.

Prop. x. Algebraically.

Let the line AB contain $2a$ linear units, of which its half AC or CB will contain a units;

and let BD contain m units.

Then the whole line and the part produced will contain $2a + m$ units,

and half the line and the part produced will contain $a + m$ units,

$$\therefore (2a + m)^2 = 4a^2 + 4am + m^2,$$

add m^2 to each of these equals,

$$\therefore (2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2.$$

Again, $(a + m)^2 = a^2 + 2am + m^2$,

add a^2 to each of these equals,

$$\therefore (a + m)^2 + a^2 = 2a^2 + 2am + m^2,$$

and doubling these equals,

$$\therefore 2(a + m)^2 + 2a^2 = 4a^2 + 4am + 2m^2.$$

But $(2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2$.

Hence $\therefore (2a + m)^2 + m^2 = 2a^2 + 2(a + m)^2$.

That is, If a number be divided into two equal parts, and the whole number and one of the parts be increased by the addition of another number, the squares of the whole number thus increased, and of the number by which it is increased, are equal to double the squares of half the number, and of half the number increased.

The algebraical results of Prop. IX, and Prop. x, are identical, (the enunciations of the two Props. arising, as in Prop. v, and Prop. VI, from the two ways of exhibiting the difference between two lines); and both may be included under the following proposition: The square on the sum of two lines and the square on their difference, are together equal to double the sum of the squares on the two lines.

Prop. XI. Two series of lines, one series decreasing and the other series increasing in magnitude, and each line divided in the same manner may be found by means of this proposition.

(1) To find the decreasing series.

In the fig. Euc. II. 11, $AB = AH + BH$,

and since $AB \,.\, BH = AH^2$, $\therefore (AH + BH) \,.\, BH = AH^2$,

$$\therefore BH^2 = AH^2 - AH \,.\, BH = AH \,.\, (AH - BH).$$

If now in HA, HL be taken equal to BH,

then $HL^2 = AH(AH - HL)$, or $AH \,.\, AL = HL^2$:

that is, AH is divided in L, so that the rectangle contained by the whole line AH and one part, is equal to the square on the other part HL. By a similar process, HL may be so divided; and so on, by always taking from the greater part of the divided line, a part equal to the less.

(2) To find the increasing series.

From the fig. it is obvious that $CF \,.\, FA = CA^2$,

Hence CF is divided in A, in the same manner as AB is divided in H, *by adding AF a line* equal to the greater segment, to the given line CA

or *AB*. And by successively adding to the last line thus divided, its greater segment, a series of lines increasing in magnitude may be found similarly divided to *AB*.

It may also be shewn that the squares on the whole line and on the less segment are equal to three times the square on the greater segment. (Euc. XIII. 4.)

To solve Prop. XI, algebraically, or to find the point H in AB such that the rectangle contained by the whole line AB and the part HB shall be equal to the square on the other part AH.

Let AB contain a linear units, and AH one of the unknown parts contain x units,

then the other part HB contains $a - x$ units.

And $\therefore a(a - x) = x^2$, by the problem,

or $x^2 + ax = a^2$, a quadratic equation.

$$\text{Whence } x = \frac{\pm a\sqrt{5} - a}{2}.$$

The former of these values of x determines the point H.

$$\text{So that } x = \frac{\sqrt{5} - 1}{2} \cdot AB = AH, \text{ one part,}$$

$$\text{and } a - x = a - AH = \frac{3 - \sqrt{5}}{2} \cdot AB = HB, \text{ the other part.}$$

It may be observed, that the parts AH and HB cannot be numerically expressed by any rational number. Approximation to their true values in terms of AB, may be made to any required degree of accuracy, by extending the extraction of the square root of 5 to any number of decimals.

To ascertain the meaning of the other result $x = -\dfrac{\sqrt{5} + 1}{2} \cdot a$.

In the equation $a(a - x) = x^2$,

for x write $-x$, then $a(a + x) = x^2$,

which when translated into words gives the following problem.

To find the length to which a given line must be produced so that the rectangle contained by the given line and the line made up of the given line and the part produced, may be equal to the square on the part produced.

Or, the problem may also be expressed as follows:

To find two lines having a given difference, such that the rectangle contained by the difference and one of them may be equal to the square on the other.

It may here be remarked, that Prop. XI. Book II, affords a simple Geometrical construction for a quadratic equation.

Prop. XII. Algebraically.

Assuming the truth of Euc. I. 47.

Let BC, CA, AB contain a, b, c linear units respectively,

and let CD, DA, contain m, n units,

then BD contains $a + m$ units.

And therefore, $c^2 = (a + m)^2 + n^2$, from the right-angled triangle ABD,

also $b^2 = m^2 + n^2$ from ACD;

$$\begin{aligned} \therefore c^2 - b^2 &= (a + m)^2 - m^2 \\ &= a^2 + 2am + m^2 - m^2 \end{aligned}$$

$$= a^2 + 2am,$$
$$\therefore c^2 = b^2 + a^2 + 2am,$$

that is, c^2 is greater than $b^2 + a^2$ by $2am$.

Prop. XIII. Case II. may be proved more simply as follows.

Since *BD* is divided into two parts in the point *D*,
therefore the squares on *CB*, *BD* are equal to twice the rectangle contained by *CB*, *BD* and the square on *CD*; (II. 7.)
add the square on *AD* to each of these equals;
therefore the squares on *CB*, *BD*, *DA* are equal to twice the rectangle *CB*, *BD*, and the squares on *CD* and *DA*,
but the squares on *BD*, *DA* are equal to the square on *AB*, (I. 47.)
and the squares on *CD*, *DA* are equal to the square on *AC*,
therefore the squares on *CB*, *BA* are equal to the square on *AC*, and twice the rectangle *CB*, *BD*. That is, &c.

Prop. XIII. Algebraically.

Let *BC*, *CA*, *AB* contain respectively a, b, c linear units, and let *BD* and *AD* also contain m and n units.

Case I. Then *DC* contains $a - m$ units.

Therefore $c^2 = n^2 + m^2$ from the right-angled triangle *ABD*,
and $b^2 = n^2 + (a - m)^2$ from *ADC*;

$$\begin{aligned} \therefore c^2 - b^2 &= m^2 - (a - m)^2 \\ &= m^2 - a^2 + 2am - m^2 \\ &= - a^2 + 2am, \end{aligned}$$
$$\therefore a^2 + c^2 = b^2 + 2am,$$
$$\text{or}\quad b^2 + 2am = a^2 + c^2,$$

that is, b^2 is less than $a^2 + c^2$ by $2am$.

Case II. $DC = m - a$ units,

$\therefore c^2 = m^2 + n^2$ from the right-angled triangle *ABD*,
and $b^2 = (m - a)^2 + n^2$ from *ACD*,

$$\begin{aligned} \therefore c^2 - b^2 &= m^2 - (m - a)^2, \\ &= m^2 - m^2 + 2am - a^2 \\ &= 2am - a^2, \end{aligned}$$
$$\therefore a^2 + c^2 = b^2 + 2am,$$
$$\text{or}\quad b^2 + 2am = a^2 + c^2,$$

that is, b^2 is less than $a^2 + c^2$ by $2am$.

Case III. Here m is equal to a.

And $b^2 + a^2 = c^2$, from the right-angled triangle *ABC*.
Add to each of these equals a^2,

$$\therefore b^2 + 2a^2 = c^2 + a^2,$$

that is, b^2 is less than $c^2 + a^2$ by $2a^2$, or $2aa$.

These two propositions, Euc. II. 12, 13, with Euc. I. 47, exhibit the relations which subsist between the sides of an obtuse-angled, an acute-angled, and right-angled triangle respectively.

NOTE ON THE ABBREVIATIONS AND ALGEBRAICAL SYMBOLS EMPLOYED IN GEOMETRY.

The ancient Geometry of the Greeks admitted no symbols besides the diagrams and ordinary language. In later times, after symbols of operation had been devised by writers on Algebra, they were very soon adopted and employed on account of their brevity and convenience, in writings purely geometrical. Dr. Barrow was one of the first who introduced algebraical symbols into the language of Elementary Geometry, and distinctly states in the preface to his Euclid, that his object is "to content the desires of those who are delighted more with symbolical than verbal demonstrations." As algebraical symbols are employed in almost all works on the mathematics, whether geometrical or not, it seems proper in this place to give some brief account of the marks which may be regarded as the alphabet of symbolical language.

The mark = was first used by Robert Recorde, in his treatise on Algebra entitled, "The Whetstone of Witte," 1557. He remarks; "And to avoide the tediouse repetition of these woordes : *is equalle to* : I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus : =, bicause noe 2 thynges can be more equalle." It was employed by him as simply affirming the equality of two numerical or algebraical expressions. Geometrical equality is not exactly the same as numerical equality, and when this symbol is used in geometrical reasonings, it must be understood as having reference to pure geometrical equality.

The signs of relative magnitude, $>$ meaning, *is greater than*, and $<$, *is less than*, were first introduced into algebra by Thomas Harriot, in his "Artis Analyticæ Praxis," which was published after his death in 1631.

The signs + and − were first employed by Michael Stifel, in his "Arithmetica Integra," which was published in 1544. The sign + was employed by him for the word *plus*, and the sign −, for the word *minus*. These signs were used by Stifel strictly as the arithmetical or algebraical signs of addition and subtraction.

The sign of multiplication × was first introduced by Oughtred in his "Clavis Mathematica," which was published in 1631. In algebraical multiplication he either connects the letters which form the factors of a product by the sign ×, or writes them as words without any sign or mark between them, as had been done before by Harriot, who first introduced the small letters to designate known and unknown quantities. However concise and convenient the notation $AB \times BC$ or $AB \, . \, BC$ may be in practice for "*the rectangle contained by the lines* AB *and* BC"; the student is cautioned against the use of it, in the early part of his geometrical studies, as its use is likely to occasion a misapprehension of Euclid's meaning, by confounding the idea of Geometrical equality with that of Arithmetical equality. Later writers on Geometry who employed the Latin language, explained the notation $AB \times BC$, by "AB *ductum in* BC"; that is, if the line AB be carried along the line BC in a normal position to it, until it come to the end C, it will then form with BC, the rectangle contained by AB and BC. Dr. Barrow sometimes expresses "*the rectangle contained by* AB *and* BC" by "*the rectangle* ABC."

Michael Stifel was the first who introduced integral exponents to denote the powers of algebraical symbols of quantity, for which he employed capital letters. Vieta afterwards used the vowels to denote known, and the *consonants*, unknown quantities, but used words to designate the powers. Simon Stevin, in his treatise on Algebra, which was published in 1605, improved the notation of Stifel, by placing the figures that indicated the powers within small circles. Peter Ramus adopted the initial letters *l*, *q*, *c*, *bq* of *latus*, *quadratus*, *cubus*, *biquadratus*, as the notation of the first four powers. Harriot exhibited the different powers of algebraical symbols by repeating the symbol, two, three, four, &c. times, according to the order of the power. Descartes restored the numerical exponents of powers, placing them at the right of the numbers, or symbols of quantity, as at the present time. Dr. Barrow employed the notation ABq, for "*the square on the line AB*," in his edition of Euclid. The notations AB^2, AB^3, for "*the square and cube on the line whose extremities are A and B*," as well as $AB \times BC$, for "*the rectangle contained by AB and BC*," are used as abbreviations in almost all works on the Mathematics, though not wholly consistent with the algebraical notations a^2 and a^3.

The symbol $\surd$, being originally the initial letter of the word *radix*, was first used by Stifel to denote the square root of the number, or of the symbol, before which it is placed.

The Hindus, in their treatises on Algebra, indicated the ratio of two numbers, or of two algebraical symbols, by placing one above the other, without any line of separation. The line was first introduced by the Arabians, from whom it passed to the Italians, and from them to the rest of Europe. This notation has been employed for the expression of geometrical ratios by almost all writers on the Mathematics, on account of its great convenience. Oughtred first used points to indicate proportion; thus, $a : b :: c : d$, means that a bears the same proportion to b, as c does to d.

QUESTIONS ON BOOK II.

1. Is *rectangle* the same as *rectus angulus?* Explain the distinction, and give the corresponding Greek terms.

2. What is meant by *the sum* of two, or more than two straight lines in Geometry?

3. Is there any difference between the straight lines by which a rectangle is said *to be contained*, and those by which *it is bounded?*

4. Define a *gnomon*. How many *gnomons* appear from the same construction in the same rectangle? Find the difference between them.

5. What axiom is assumed in proving the first eight propositions of the Second Book of Euclid?

6. Of equal squares and equal rectangles, which must *necessarily* coincide?

7. How may a rectangle be dissected so as to form an equivalent rectangle of any proposed length?

8. When the adjacent sides of a rectangle are commensurable, the area of the rectangle is properly represented by the product of the number of units in two adjacent sides of the rectangle. Illustrate this by considering the case when the two adjacent sides contain 3 and 4 units respectively, and distinguish between the units of the factors and the units of the product. Shew generally that a rectangle whose adjacent sides are represented by the integers a and b, is represented by ab. Also shew, that in the same sense, *the rectangle is represented by* $\dfrac{ab}{mn}$, if the sides be represented by $\dfrac{a}{m}$, $\dfrac{b}{n}$.

9. Why may not Algebraical or Arithmetical proofs be substituted (as being shorter) for the demonstrations of the Propositions in the Second Book of Euclid?

10. In what sense is the *area* of a triangle said to be equal to half the product of its base and its altitude? What two propositions of Euclid may be adduced to prove it?

11. How do you shew that the area of a rhombus is equal to half the rectangle contained by the diagonals?

12. How may a rule be deduced for finding a numerical expression for the area of any parallelogram, when two adjacent sides are given?

13. The area of a trapezium which has two of its sides parallel is equal to that of a rectangle contained by its altitude and half the sum of its parallel sides. What propositions of the First and Second Books of Euclid are employed to prove this? Of what service is the above in the mensuration of fields with irregular borders?

14. From what propositions of Euclid may be deduced the following rule for finding the area of any quadrilateral figure:—"Multiply the sum of the perpendiculars drawn from opposite angles of the figure upon the diagonal joining the other two angles, and take half the product."

15. In Euclid, II. 3, where must be the point of division of the line, so that the rectangle contained by the two parts may be a maximum? Exemplify in the case where the line is 12 inches long.

16. How may the demonstration of Euclid II. 4, be legitimately shortened? Give the Algebraical proof, and state on what suppositions it can be regarded as a proof.

17. Shew that the proof of Euc. II. 4, can be deduced from the two previous propositions without any geometrical construction.

18. Shew that if the two complements be together equal to the two squares, the given line is bisected.

19. If the line AB, as in Euc. II. 4, be divided into any three parts, enunciate and prove the analogous proposition.

20. Prove geometrically that if a straight line be trisected, the square on the whole line equals nine times the square on a third part of it.

21. Deduce from Euc. II. 4, a proof of Euc. I. 47.

22. If a straight line be divided into two parts, when is the rectangle contained by the parts, *the greatest possible?* and when is the sum of the squares of the parts, *the least possible?*

23. Shew that if a line be divided into two equal parts and into two unequal parts; the part of the line between the points of section is equal to half the difference of the unequal parts.

24. If half the sum of two unequal lines be increased by half their difference, the sum will be equal to the greater line: and if the sum of two lines be diminished by half their difference, the remainder will be equal to the less line.

25. Explain what is meant by the *internal* and *external segments* of a line; and show that the sum of the external segments of a line or the difference of the internal segments is double the distance between the points of section and bisection of the line.

26. Shew how Euc. II. 6, may be deduced immediately from the preceding Proposition.

27. Prove Geometrically that the squares on the sum and difference of two lines are equal to twice the squares on the lines themselves.

28. A given rectangle is divided by two straight lines into four rectangles. *Given the* areas of the two which have not common sides: find *the areas of the* other two.

29. In how many ways may the difference of two lines be exhibited? Enunciate the propositions in Book II. which depend on that circumstance.

30. How may a series of lines be found similarly divided to the line AB in Euc. II. 11?

31. Divide Algebraically a given line (a) into two parts, such that the rectangle contained by the whole and one part may be equal to the square of the other part. Deduce Euclid's construction from one solution, and explain the other.

32. Given the lesser segment of a line, divided as in Euc. II. 11, find the greater.

33. Enunciate the Arithmetical theorems expressed by the following Algebraical formulæ,

$$(a+b)^2 = a^2 + 2ab + b^2 : a^2 - b^2 = (a+b)\,(a-b) : (a-b)^2 = a^2 - 2ab + b^2,$$

and state the corresponding Geometrical propositions.

34. Shew that the first of the Algebraical propositions,

$$(a+x)\,(a-x) + x^2 = a^2 : (a+x)^2 + (a-x)^2 = 2a^2 + 2x^2,$$

is equivalent to the two propositions v. and vi., and the second of them, to the two propositions ix. and x. of the Second Book of Euclid.

35. Prove Euc. II. 12, when the perpendicular BE is drawn from B on AC produced to E, and shew that the rectangle BC, CD is equal to the rectangle AC, CE.

36. Include the first two cases of Euc. II. 13, in one proof.

37. In the second case of Euc. II. 13, draw a perpendicular CE from the obtuse angle C upon the side AB, and prove that the square on AB is equal to the rectangle AB, AE together with the rectangle BC, BD.

38. Enunciate Euc. II. 13, and give an Algebraical or Arithmetical proof of it.

39. The sides of a triangle are as 3, 4, 5. Determine whether the angles between 3, 4; 4, 5; and 3, 5; respectively are greater than, equal to, or less than, a right angle.

40. Two sides of a triangle are 4 and 5 inches in length, if the third side be $6\frac{5}{16}$ inches, the triangle is acute-angled, but if it be $6\frac{7}{16}$ inches, the triangle is obtuse-angled.

41. A triangle has its sides 7, 8, 9 units respectively; a strip of breadth 2 units being taken off all round from the triangle, find the area of the remainder.

42. If the original figure, Euc. II. 14, were a right-angled triangle, whose *sides* were represented by 8 and 9, what number would represent the side of a square of the same area? Shew that the perimeter of the square is less than the perimeter of the triangle.

43. If the sides of a rectangle are 8 feet and 2 feet, what is the side of the equivalent square?

44. "All plane rectilineal figures admit of quadrature." Point out the succession of steps by which Euclid establishes the truth of this proposition.

45. Explain the construction (without proof) for making a square equal to a plane polygon.

46. Shew from Euc. II. 14, that any algebraical surd as $\sqrt{a}$ can be represented by a line, if the unit be a line.

47. Could any of the *propositions* of the Second Book be made *corollaries* to other propositions, with advantage? Point out any such propositions, and give your reasons for the alterations you would make.

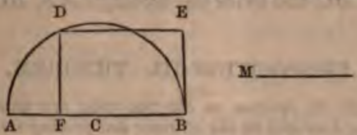
GEOMETRICAL EXERCISES ON BOOK II.

PROPOSITION I. PROBLEM.

Divide a given straight line into two parts such, that their rectangle may be equal to a given square; and determine the greatest square which the rectangle can equal.

Let AB be the given straight line, and let M be the side of the given square.

It is required to divide the line AB into two parts, so that the rectangle contained by them may be equal to the square on M.



Bisect AB in C, with center C, and radius CA or CB, describe the semicircle ADB.

At the point B draw BE at right angles to AB and equal to M.

Through E, draw ED parallel to AB and cutting the semicircle in D;

and draw DF parallel to EB meeting AB in F.

Then AB is divided in F, so that the rectangle AF, FB is equal to the square on M. (II. 14.)

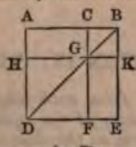
The square will be the greatest, when ED touches the semicircle, or when M is equal to half of the given line AB.

PROPOSITION II. THEOREM.

The square on the excess of one straight line above another is less than the squares on the two lines by twice their rectangle.

Let AB, BC be the two straight lines, whose difference is AC.

Then the square on AC is less than the squares on AB and BC by twice the rectangle contained by AB and BC.



Constructing as in Prop. 4. Book II.

Because the complement AG is equal to GE,

add to each CK,

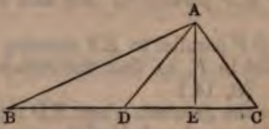
therefore the whole AK is equal to the whole CE;

and AK, CE together are double of AK:
but AK, CE are the gnomon AKF and CK,
and AK is the rectangle contained by AB, BC;
therefore the gnomon AKF and CK,
are equal to twice the rectangle AB, BC,
but AE, CK are equal to the squares on AB, BC;
taking the former equals from these equals,
therefore the difference of AE and the gnomon AKF is equal to the difference between the squares on AB, BC, and twice the rectangle AB, BC;
but the difference AE and the gnomon AKF is the figure HF, which is equal to the square on AC.
Wherefore the square on AC is equal to the difference between the squares on AB, BC, and twice the rectangle AB, BC.

PROPOSITION III. THEOREM.

In any triangle the squares on the two sides are together double of the squares on half the base and on the straight line joining its bisection with the opposite angle.

Let ABC be a triangle, and AD the line drawn from the vertex A to the bisection D of the base BC.

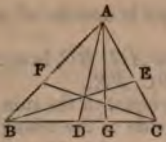


From A draw AE perpendicular to BC.
Then, in the obtuse-angled triangle ABD, (II. 12.);
the square on AB exceeds the squares on AD, DB, by twice the rectangle BD, DE:
and in the acute-angled triangle ADC, (II. 13.);
the square on AC is less than the squares on AD, DC, by twice the rectangle CD, DE:
wherefore, since the rectangle BD, DE is equal to the rectangle CD, DE; it follows that the squares on AB, AC are double of the squares on AD, DB.

PROPOSITION IV. THEOREM.

If straight lines be drawn from each angle of a triangle bisecting the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.

Let ABC be any triangle, and let AD, BE, CF be drawn from A, B, C to D, E, F, the bisections of the opposite sides of the triangle: draw AG perpendicular to BC.



Then the square on AB is equal to the squares on BD, DA together with twice the rectangle BD, DG, (II. 12.)

and the square on AC is equal to the squares on CD, DA diminished by twice the rectangle CD, DG; (II. 13.)

therefore the squares on AB, AC are equal to twice the square on BD, and twice the square on AD; for DC is equal to BD:

and twice the squares on AB, AC are equal to the square on BC, and four times the square on AD: for BC is twice BD.

Similarly, twice the squares on AB, BC are equal to the square on AC, and four times the square on BE:

also twice the squares on BC, CA are equal to the square on AB, and four times the square on FC:

hence, by adding these equals,

four times the squares on AB, AC, BC are equal to four times the squares on AD, BE, CF together with the squares on AB, AC, BC:

and taking the squares on AB, AC, BC from these equals,

therefore three times the squares on AB, AC, BC are equal to four times the squares on AD, BE, CF.

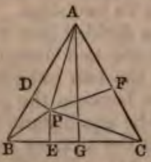
PROPOSITION V. THEOREM.

The sum of the perpendiculars let fall from any point within an equilateral triangle, will be equal to the perpendicular let fall from one of its angles upon the opposite side. Is this proposition true when the point is in one of the sides of the triangle? In what manner must the proposition be enunciated when the point is without the triangle?

Let ABC be an equilateral triangle, and P any point within it:

and from P let fall PD, PE, PF perpendiculars on the sides AB, BC, CA respectively, also from A let fall AG perpendicular on the base BC.

Then AG is equal to the sum of PD, PE, PF.



From P draw PA, PB, PC to the angles A, B, C.

Then the triangle ABC is equal to the three triangles PAB, PBC, PCA.

But since every rectangle is double of a triangle of the same base and altitude, (I. 41.)

therefore the rectangle AG, BC, is equal to the three rectangles AB, PD; AC, PF and BC, PE.

Whence the line AG is equal to the sum of the lines PD, PE, PF.

If the point P fall on one side of the triangle, or coincide with E:

then the triangle ABC is equal to the two triangles APC, BPA: whence AG is equal to the sum of the two perpendiculars PD, PF.

If the point P fall without the base BC of the triangle:

then the triangle ABC is equal to the difference between the sum of the two triangles APC, BPA, and the triangle PCB.

Whence AG is equal to the difference between the sum of PD, PF, and PE.

I.

6. If the straight line AB be divided into two unequal parts in D, and into two unequal parts in E, the rectangle contained by AE, EB, will be greater or less than the rectangle contained by AD, DB, according as E is nearer to, or further from, the middle point of AB, than D.

7. Produce a given straight line in such a manner that the square on the whole line thus produced, shall be equal to twice the square on the given line.

8. If AB be the line so divided in the points C and D, (fig. Euc. II. 5.) shew that $AB^2 = 4.CD^2 + 4.AD.DB$.

9. Divide a straight line into two parts, such that the sum of their squares may be the least possible.

10. Divide a line into two parts, such that the sum of their squares shall be double the square on another line.

11. Shew that the difference between the squares on the two unequal parts (fig. Euc. II. 9.) is equal to twice the rectangle contained by the whole line, and the part between the points of section.

12. Shew how in all the possible cases, a straight line may be *geometrically* divided into two such parts, that the sum of their squares shall be equal to a given square.

13. Divide a given straight line into two parts, such that the squares on the whole line and on one of the parts shall be equal to twice the square on the other part.

14. Any rectangle is the half of the rectangle contained by the diameters of the squares on its two sides.

15. If a straight line be divided into two equal and into two unequal parts, the squares on the two unequal parts are equal to twice the rectangle contained by the two unequal parts, together with four times the square on the line between the points of section.

16. If the points C, D be equidistant from the extremities of the straight line AB, shew that the squares constructed on AD and AC, exceed twice the rectangle AC, AD by the square constructed on CD.

17. If any point be taken in the plane of a parallelogram from which perpendiculars are let fall on the diagonal, and on the sides which include it, the rectangle of the diagonal and the perpendicular

on it, is equal to the sum or difference of the rectangles of the sides and the perpendiculars on them.

18. *ABCD* is a rectangular parallelogram, of which *A*, *C* are opposite angles, *E* any point in *BC*, *F* any point in *CD*. Prove that twice the area of the triangle *AEF* together with the rectangle *BE*, *DF* is equal to the parallelogram *AC*.

II.

19. Shew how to produce a given line, so that the rectangle contained by the whole line thus produced, and the produced part, shall be equal to the square (1) on the given line (2) on the part produced.

20. If in the figure Euc. II. 11, we join *BF* and *CH*, and produce *CH* to meet *BF* in *L*, *CL* is perpendicular to *BF*.

21. If a line be divided, as in Euc. II. 11, the squares on the whole line and one of the parts are together three times the square on the other part.

22. If in the fig. Euc. II. 11, the points *F*, *D* be joined cutting *AHB*, *GHK* in *f*, *d* respectively; then shall $Ff = Dd$.

III.

23. If from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, the squares on the distances between the angles and the common intersection, are together one-third of the squares on the sides of the triangle.

24. *ABC* is a triangle of which the angle at *C* is obtuse, and the angle at *B* is half a right angle: *D* is the middle point of *AB*, and *CE* is drawn perpendicular to *AB*. Shew that the square on *AC* is double of the squares on *AD* and *DE*.

25. If an angle of a triangle be two-thirds of two right angles, shew that the square on the side subtending that angle is equal to the squares on the sides containing it, together with the rectangle contained by those sides.

26. The square described on a straight line drawn from one of the angles at the base of a triangle to the middle point of the opposite side, is equal to the sum or difference of the square on half the side bisected, and the rectangle contained between the base and that part of it, or of it produced, which is intercepted between the same angle and a perpendicular drawn from the vertex.

27. *ABC* is a triangle of which the angle at *C* is obtuse, and the angle at *B* is half a right angle: *D* is the middle point of *AB*, and *CE* is drawn perpendicular to *AB*. Shew that the square on *AC* is double of the squares on *AD* and *DE*.

28. Produce one side of a scalene triangle, so that the rectangle under it and the produced part may be equal to the difference of the squares on the other two sides.

29. Given the base of any triangle, the area, and the line bisecting the base, construct the triangle.

IV.

30. Shew that the square on the hypotenuse of a right-angled triangle, is equal to four times the area of the triangle together with the square on the difference of the sides.

31. In the triangle ABC, if AD be the perpendicular let fall upon the side BC; then the square on AC together with the rectangle contained by BC, BD is equal to the square on AB together with the rectangle CB, CD.

32. ABC is a triangle, right angled at C, and CD is the perpendicular let fall from C upon AB; if HK is equal to the sum of the sides AC, CB, and LM to the sum of AB, CD, shew that the square on HK together with the square on CD is equal to the square on LM.

33. ABC is a triangle having the angle at B a right angle: it is required to find in AB a point P such that the square on AC may exceed the squares on AP and PC by half the square on AB.

34. In a right-angled triangle, the square on that side which is the greater of the two sides containing the right angle, is equal to the rectangle by the sum and difference of the other sides.

35. The hypotenuse AB of a right-angled triangle ABC is trisected in the points D, E; prove that if CD, CE be joined, the sum of the squares on the sides of the triangle CDE is equal to two-thirds of the square on AB.

36. From the hypotenuse of a right-angled triangle portions are cut off equal to the adjacent sides: shew that the square on the middle segment is equivalent to twice the rectangle under the extreme segments.

V.

37. Prove that the square on any straight line drawn from the vertex of an isosceles triangle to the base, is less than the square on a side of the triangle by the rectangle contained by the segments of the base: and conversely.

38. If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, the rectangle contained by that side and the segment of it intercepted between the perpendicular and base, is equal to the half of the square described upon the base.

39. If in an isosceles triangle a perpendicular be let fall from one of the equal angles to the opposite side, the square on the perpendicular is equal to the square on the line intercepted between the other equal angle and the perpendicular, together with twice the rectangle contained by the segments of that side.

40. The square on the base of an isosceles triangle whose vertical angle is a right angle, is equal to four times the area of the triangle.

41. Describe an isosceles obtuse-angled triangle, such that the square on the side subtending the obtuse angle may be three times the square on either of the sides containing the obtuse angle.

42. If AB, one of the sides of an isosceles triangle ABC be produced beyond the base to D, so that $BD = AB$, shew that

$$CD^2 = AB^2 + 2 . BC^2.$$

43. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC, and EB be joined; prove that $BE^2 = BC \times DE + CE^2$.

44. If ABC be an isosceles triangle of which the angles at B and C are each double of A; then the square on AC is equal to the square on BC together with the rectangle contained by AC and BC.

VI.

45. Shew that in a parallelogram the squares on the diagonals are equal to the sum of the squares on all the sides.

46. If $ABCD$ be any rectangle, A and C being opposite angles, and O any point either within or without the rectangle:

$$OA^2 + OC^2 = OB^2 + OD^2.$$

47. In any quadrilateral figure, the sum of the squares on the diagonals together with four times the square on the line joining their middle points, is equal to the sum of the squares on all the sides.

48. In any trapezium, if the opposite sides be bisected, the sum of the squares on the other two sides, together with the squares on the diagonals, is equal to the sum of the squares on the bisected sides, together with four times the square on the line joining the points of bisection.

49. The squares on the diagonals of a trapezium are together double the squares on the two lines joining the bisections of the opposite sides.

50. In any trapezium two of whose sides are parallel, the squares on the diagonals are together equal to the squares on its two sides which are not parallel, and twice the rectangle contained by the sides which are parallel.

51. If the two sides of a trapezium be parallel, shew that its area is equal to that of a rectangle contained by its altitude and half the sum of the parallel sides.

52. If a trapezium have two sides parallel, and the other two equal, shew that the rectangle contained by the two parallel sides, together with the square on one of the other sides, will be equal to the square on the straight line joining two opposite angles of the trapezium.

53. If squares be described on the sides of any triangle and the angular points of the squares be joined; the sum of the squares on the sides of the hexagonal figure thus formed is equal to four times the sum of the squares on the sides of the triangle.

VII.

54. Find the side of a square equal to a given equilateral triangle.

55. Find a square which shall be equal to the sum of two given rectilineal figures.

56. To divide a given straight line so that the rectangle under its segments may be equal to a given rectangle.

57. Construct a rectangle equal to a given square and having the difference of its sides equal to a given straight line.

58. Shew how to describe a rectangle equal to a given square, and having one of its sides equal to a given straight line.

GEOMETRICAL EXERCISES ON BOOK I.

HINTS, &c.

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8. This is a particular case of Euc. i. 22. The triangle however may be described by means of Euc. i. 1. Let AB be the given base, produce AB both ways to meet the circles in D, E (fig. Euc. i. 1.); with center A, and radius AE, describe a circle, and with center B and radius BD, describe another circle cutting the former in G. Join GA, GB.

9. Apply Euc. i. 6, 8.

10. This is proved by Euc. i. 32, 13, 5.

11. Let fall also a perpendicular from the vertex on the base.

12. Apply Euc. i. 4.

13. Let CAB be the triangle (fig. Euc. i. 10.) CD the line bisecting the angle ACD and the base AB. Produce CD, and make DE equal to CD, and join AE. Then CB may be proved equal to AE, also AE to AC.

14. Let AB be the given line, and C, D the given points. From C draw CE perpendicular to AB, and produce it making EF equal to CE, join FD, and produce it to meet the given line in G, which will be the point required.

15. Make the construction as the enunciation directs, then by Euc. i. 4, BH is proved equal to CK: and by Euc. i. 13, 6, OB is shewn to be equal to OC.

16. This proposition requires for its proof the case of equal triangles omitted in Euclid:—namely, when two sides and one angle are given, but not the angle included by the given sides.

17. The angle BCD may be shewn to be equal to the sum of the angles ABC, ADC.

18. The angles ADE, AED may be each proved to be equal to the complements of the angles at the base of the triangle.

19. The angles CAB, CBA, being equal, the angles CAD, CBE are equal, Euc. i. 13. Then, by Euc. i. 4, CD is proved to be equal to CE. And by Euc. i. 5, 32, the angle at the vertex is shewn to be four times either of the angles at the base.

20. Let AB, CD be two straight lines intersecting each other in E, and let P be the given point, within the angle AED. Draw EF bisecting the angle AED, and through P draw PGH parallel to EF, and cutting ED, EB in G, H. Then EG is equal to EH. And by bisecting the angle DEB and drawing through P a line parallel to this line, another solution is obtained. It will be found that the two lines are at right angles to each other.

21. Let the two given straight lines meet in A, and let P be the given point. Let PQR be the line required, meeting the lines AQ, AR in Q and R, so that PQ is equal to QR. Through P draw PS parallel to AR and join RS. Then APSR is a parallelogram and AS, PR the diagonals. Hence the construction.

22. Let the two straight lines AB, AC meet in A. In AB take any point D, and from AC cut off AE equal to AD, and join DE. On DE, or DE produced, take DF equal to the given line, and through F draw FG parallel to AB meeting AC in G, and through G draw GH parallel to DE meeting AB in H. Then GH is the line required.

23. The two given points may be both on the same side, or one point may be on each side of the line. If the point required in the line be supposed to be found, and lines be drawn joining this point and the given points, an isosceles triangle is formed, and if a perpendicular be drawn on the base from the point in the line: the construction is obvious.

24. The problem is simply this—to find a point in one side of a triangle from which the perpendiculars drawn to the other two sides shall be equal. If all the positions of these lines be considered, it will readily be seen in what case the problem is impossible.

25. If the *isosceles triangle* be obtuse-angled, by Euc. I. 5, 32, the truth will be made evident. If the triangle be acute-angled, the enunciation of the proposition requires some modification.

26. Construct the figure and apply Euc. I. 5, 32, 15.

If the isosceles triangle have its vertical angle less than two-thirds of a right-angle, the line ED produced, meets AB produced towards the base, and then 3 . AEF = 4 right angles + AFE. If the vertical angle be greater than two-thirds of a right angle, ED produced meets AB produced towards the vertex, then 3 . AEF = 2 right angles + AFE.

27. Let ABC be an isosceles triangle, and from any point D in the base BC, and the extremity B, let three lines DE, DF, BG be drawn to the sides and making equal angles with the base. Produce ED and make DH equal to DF and join BH.

28. In the isosceles triangle ABC, let the line DFE which meets the side AC in D and AB produced in E, be bisected by the base in the point E. Then DC may be shewn to be equal to BE.

29. If two equal straight lines be drawn terminated by two lines which meet in a point, they will cut off triangles of equal area. Hence the two triangles have a common vertical angle and their areas and bases equal. By Euc. I. 32 it is shewn that the angle contained by the bisecting lines is equal to the exterior angle at the base.

30. There is an omission in this question. After the words "making equal angles with the sides," add, "and be equal to each other respectively." (1), (3) Apply Euc. I. 26, 4. (2) The equal lines which bisect the sides may be shewn to make equal angles with the sides.

31. At C make the angle BCD equal to the angle ACB, and produce AB to meet CD in D.

32. By bisecting the hypotenuse, and drawing a line from the vertex to the point of bisection, it may be shewn that this line forms with the shorter side and half the hypotenuse an isosceles triangle.

33. Let ABC be a triangle, having the right angle at A, and the angle at C greater than the angle at B, also let AD be perpendicular to the base, and AE be the line drawn to E the bisection of the base. Then AE may be proved equal to BE or EC independently of Euc. III. 31.

34. Produce EG, FG to meet the perpendiculars CE, BF, produced if necessary. The demonstration is obvious.

35. If the given triangle have both of the angles at the base, acute angles; the difference of the angles at the base is at once obvious from Euc. I. 32. If one of the angles at the base be obtuse, does the property hold good?

36. Let ABC be a triangle having the angle ACB double of the angle ABC, and let the perpendicular AD be drawn to the base BC. Take DE equal to DC and join AE. Then AE may be proved to be equal to EB.

If ACB be an obtuse angle, then AC is equal to the sum of the segments of the base, made by the perpendicular from the vertex A.

37. Let the sides AB, AC of any triangle ABC be produced, the ex-

terior angles bisected by two lines which meet in D, and let AD be joined, then AD bisects the angle BAC. For draw DE perpendicular on BC, also DF, DG perpendiculars on AB, AC produced, if necessary. Then DF may be proved equal to DG, and the squares on DF, DA are equal to the squares on FG, GA, of which the square on FD is equal to the square on DG; hence AF is equal to AG, and Euc. I. 8, the angle BAC is bisected by AD.

38. The line required will be found to be equal to half the sum of the two sides of the triangle.

39. Apply Euc. I. 1, 9.

40. The angle to be trisected is one-fourth of a right angle. If an equilateral triangle be described on one of the sides of a triangle which contains the given angle, and a line be drawn to bisect that angle of the equilateral triangle which is at the given angle, the angle contained between this line and the other side of the triangle will be one-twelfth of a right angle, or equal to one-third of the given angle.

It may be remarked, generally, that any angle which is the half, fourth, eighth, &c. part of a right angle, may be trisected by Plane Geometry.

41. Apply Euc. I. 20.

42. Let ABC, DBC be two equal triangles on the same base, of which ABC is isosceles, fig. Euc. I. 37. By producing AB and making AG equal to AB or AC, and joining GD, the perimeter of the triangle ABC may be shewn to be less than the perimeter of the triangle DBC.

43. Apply Euc. I. 20.

44. For the first case, see Theo. 32, p. 76: for the other two cases, apply Euc. I. 19.

45. This is obvious from Euc. I. 26.

46. By Euc. I. 29, 6, FC may be shewn equal to each of the lines EF, FG.

47. Join GA and AF, and prove GA and AF to be in the same straight line.

48. Let the straight line drawn through D parallel to BC meet the side AB in E, and AC in F. Then in the triangle EBD, EB is equal to ED, by Euc. I. 29, 6. Also, in the triangle EAD, the angle EAD may be shewn equal to the angle EDA, whence EA is equal to ED, and therefore AB is bisected in E. In a similar way it may be shewn, by bisecting the angle C, that AC is bisected in F. Or the bisection of AC in F may be proved when AB is shewn to be bisected in E.

49. The triangle formed will be found to have its sides respectively parallel to the sides of the original triangle.

50. If a line equal to the given line be drawn from the point where the two lines meet, and parallel to the other given line; a parallelogram may be formed, and the construction effected.

51. Let ABC be the triangle; AD perpendicular to BC, AE drawn to the bisection of BC, and AF bisecting the angle BAC. Produce AD and make DA′ equal to AD: join FA′, EA′.

52. If the point in the base be supposed to be determined, and lines drawn from it parallel to the sides, it will be found to be in the line which bisects the vertical angle of the triangle.

53. Let ABC be the triangle, at C draw CD perpendicular to CB and equal to the sum of the required lines, through D draw DE parallel to CB *meeting AC in* E, and draw EF parallel to DC, meeting BC in F. Then *EF is equal to DC.* Next produce CB, making CG equal to CE, and join *EG cutting AB* in H. From H draw HK perpendicular to EAC, and

HL perpendicular to BC. Then HK and HL together are equal to DC, The proof depends on Theorem 27, p. 75.

54. Let C′ be the intersection of the circles on the other side of the base, and join AC′, BC′. Then the angles CBA, C′BA being equal, the angles CBP, C′BP are also equal, Euc. I. 13: next by Euc. I. 4, CP, PC′ are proved equal; lastly prove CC′ to be equal to CP or PC′.

55. In the fig. Euc. I. 1, produce AB both ways to meet the circles in D and E, join CD, CE, then CDE is an isosceles triangle, having each of the angles at the base one-fourth of the angle at the vertex. At E draw EG perpendicular to DB and meeting DC produced in G. Then CEG is an equilateral triangle.

56. Join CC′, and shew that the angles CC′F, CC′G are equal to two right angles; also that the line FC′G is equal to the diameter.

57. Construct the figure and by Euc. I. 32. If the angle BAC be a right angle, then the angle BDC is half a right angle.

58. Let the lines which bisect the three exterior angles of the triangle ABC form a new triangle A′B′C′. Then each of the angles at A′, B′, C′ may be shewn to be equal to half of the angles at A and B, B and C, C and A respectively. And it will be found that half the sums of every two of three unequal numbers whose sum is constant, have less differences than the three numbers themselves.

59. The first case may be shewn by Euc. I. 4: and the second by Euc. I. 32, 6, 15.

60. At D any point in a line EF, draw DC perpendicular to EF and equal to the given perpendicular on the hypotenuse. With centre C and radius equal to the given base describe a circle cutting EF in B. At C draw CA perpendicular to CB and meeting EF in A. Then ABC is the triangle required.

61. Let ABC be the required triangle having the angle ACB a right angle. In BC produced, take CE equal to AC, and with center B and radius BA describe a circular arc cutting CE in D, and join AD. Then DE is the difference between the sum of the two sides AC, CB and the hypotenuse AB; also one side AC the perpendicular is given. Hence the construction. On any line EB take EC equal to the given side, ED equal to the given difference. At C, draw CA perpendicular to CB, and equal to EC, join AD, at A in AD make the angle DAB equal to ADB, and let AB meet EB in B. Then ABC is the triangle required.

62. (1) Let ABC be the triangle required, having ACB the right angle. Produce AB to D making AD equal to AC or CB: then BD is the sum of the sides. Join DC: then the angle ADC is one-fourth of a right angle, and DBC is one-half of a right angle. Hence to construct: at B in BD make the angle DBM equal to half a right angle, and at D the angle BDC equal to one-fourth of a right angle, and let DC meet BM in C. At C draw CA at right angles to BC meeting BD in A: and ABC is the triangle required.

(2) Let ABC be the triangle, C the right angle: from AB cut off AD equal to AC; then BD is the difference of the hypotenuse and one side. Join CD; then the angles ACD, ADC are equal, and each is half the supplement of DAC, which is half a right angle. Hence the construction.

63. Take any straight line terminated at A. Make AB equal to the difference of the sides, and AC equal to the hypotenuse. At B *make the* angle CBD equal to half a right angle, and with center A *and radius* AC describe a circle cutting BD in D: join AD, and draw *DE perpendicular to* AC. Then ADE is the requiredtriangle.

64. Let BC the given base be bisected in D. At D draw DE at right angles to BC and equal to the sum of one side of the triangle and the perpendicular from the vertex on the base: join DB, and at B in BE make the angle EBA equal to the angle BED, and let BA meet DE in A: join AC, and ABC is the isosceles triangle.

65. This construction may be effected by means of Prob. 4, p. 71.

66. The perpendicular from the vertex on the base of an equilateral triangle bisects the angle at the vertex which is two-thirds of one right angle.

67. Let ABC be the equilateral triangle of which a side is required to be found, having given BD, CD the lines bisecting the angles at B, C. Since the angles DBC, DCB are equal, each being one-third of a right angle, the sides BD, DC are equal, and BDC is an isosceles triangle having the angle at the vertex the supplement of a third of two right angles. Hence the side BC may be found.

68. Let the given angle be taken, (1) as the *included angle* between the given sides; and (2) as the *opposite angle* to one of the given sides. In the latter case, an ambiguity will arise if the angle be an acute angle, and opposite to the less of the two given sides.

69. Let ABC be the required triangle, BC the given base, CD the given difference of the sides AB, AC: join BD, then DBC by Euc. I. 18, can be shewn to be half the difference of the angles at the base, and AB is equal to AD. Hence at B in the given base BC, make the angle CBD equal to half the difference of the angles at the base. On CB take CE equal to the difference of the sides, and with center C and radius CE, describe a circle cutting BD in D: join CD and produce it to A, making DA equal to DB. Then ABC is the triangle required.

70. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Then bisect the angles at the base; and from the point where these lines meet, draw lines parallel to the sides and meeting the base.

71. Let ABC be the required triangle, BC the given base, and the side AB greater than AC. Make AD equal to AC, and draw CD. Then the angle BCD may be shewn to be equal to half the difference, and the angle DCA equal to half the sum of the angles at the base. Hence ABC, ACB the angles at the base of the triangle are known.

72. Let the two given lines meet in A, and let B be the given point. If BC, BD be supposed to be drawn making equal angles with AC, and if AD and DC be joined, BCD is the triangle required, and the figure ACBD may be shewn to be a parallelogram. Whence the construction.

73. It can be shewn that lines drawn from the angles of a triangle to bisect the opposite sides, intersect each other at a point which is two-thirds of their lengths from the angular points from which they are drawn. Let ABC be the triangle required, AD, BE, CF the given lines from the angles drawn to the bisections of the opposite sides and intersecting in G. Produce GD, making DH equal to DG, and join BH, CH: the figure GBHC is a parallelogram. Hence the construction.

74. Let ABC (fig. to Euc. I. 20.) be the required triangle, having the base BC equal to the given base, the angle ABC equal to the given angle, and the two sides BA, AC together equal to the given line BD. *Join DC, then* since AD is equal to AC, the triangle ACD is isosceles, *and therefore* the angle ADC is equal to the angle ACD. Hence the *construction.*

75. Let ABC be the required triangle (fig. to Euc. I. 18), having the *angle ACB* equal to the given angle, and the base BC equal to the given line, also CD equal to the difference of the two sides AB, AC. If BD be joined, then ABD is an isosceles triangle. Hence the synthesis. Does this construction hold good in all cases?

76. Let ABC be the required triangle, (fig. Euc. I. 18), of which the side BC is given and the angle BAC, also CD the difference between the sides AB, AC. Join BD; then AB is equal to AD, because CD is their difference, and the triangle ABD is isosceles, whence the angle ABD is equal to the angle ADB; and since BAD and twice the angle ABD are equal to two right angles, it follows that ABD is half the supplement of the given angle BAC. Hence the construction of the triangle.

77. Let AB be the given base: at A draw the line AD to which the line bisecting the vertical angle is to be parallel. At B draw BE parallel to AD; from A draw AE equal to the given sum of the two sides to meet BE in E. At B make the angle EBC equal to the angle BEA, and draw CF parallel to AD. Then ACB is the triangle required.

78. Take any point in the given line, and apply Euc. I. 23, 31.

79. On one of the parallel lines take EF equal to the given line, and with center E and radius EF describe a circle cutting the other in G. Join EG, and through A draw ABC parallel to EG.

80. This will appear from Euc. I. 29, 15, 26.

81. Let AB, AC, AD, be the three lines. Take any point E in AC, and on EC make EF equal to EA. through F draw FG parallel to AB, join GE and produce it to meet AB in H. Then GE is equal to GH.

82. Apply Euc. I. 32, 29.

83. From E draw EG perpendicular on the base of the triangle, then ED and EF may each be proved equal to EG, and the figure shewn to be equilateral. Three of the angles of the figure are right angles.

84. The greatest parallelogram which can be constructed with given sides can be proved to be rectangular.

85. Let AB be one of the diagonals: at A in AB make the angle BAC less than the required angle, and at A in AC make the angle CAD equal to the required angle. Bisect AB in E and with center E and radius equal to half the other diagonal describe a circle cutting AC, AD in F, G. Join FB, BG: then AFBG is the parallelogram required.

86. This problem is the same as the following; having given the base of a triangle, the vertical angle and the sum of the sides, to construct the triangle. This triangle is one half of the required parallelogram.

87. Draw a line AB equal to the given diagonal, and at the point A make an angle BAC equal to the given angle. Bisect AB in D, and through D draw a line parallel to the given line and meeting AC in C. This will be the position of the other diagonal. Through B draw BE parallel to CA, meeting CD produced in E; join AE, and BC. Then ACBE is the parallelogram required.

88. Construct the figures and by Euc. I. 24.

89. By Euc. I. 4, the opposite sides may be proved to be equal.

90. Let ABCD be the given parallelogram; construct the other parallelogram A′B′C′D′ by drawing the lines required, also the diagonals AC, A′C′, and shew that the triangles ABC, A′B′C′ are equiangular.

91. A′D′ and B′C′ may be proved to be parallel.

92. Apply Euc. I. 29, 32.

93. The points D, D′, are the intersections of the diagonals of two rectangles; if the rectangles be completed, and the lines OD, OD′ be produced, they will be the other two diagonals.

94. Let the line drawn from A fall without the parallelogram, and

let CC′, BB′, DD′, be the perpendiculars from C, B, D, on the line drawn from A; from B draw BE parallel to AC′, and the truth is manifest. Next, let the line from A be drawn so as to fall within the parallelogram.

95. Let the diagonals intersect in E. In the triangles DCB, CDA, two angles in each are respectively equal and one side DE: wherefore the diagonals DB, AC are equal: also since DE, EC are equal, it follows that EA, EB are equal. Hence DEC, AEB are two isosceles triangles having their vertical angles equal, wherefore the angles at their bases are equal respectively, and therefore the angle CDB is equal to DBA.

96. (1) By supposing the point P found in the side AB of the parallelogram ABCD, such that the angle contained by AP, PC may be bisected by the line PD; CP may be proved equal to CD; hence the solution.

(2) By supposing the point P found in the side AB produced, so that PD may bisect the angle contained by ABP and PC; it may be shewn that the side AB must be produced, so that BP is equal to BD.

97. This may be shewn by Euc. I. 35.

98. Let D, E, F be the bisections of the sides AB, BC, CA of the triangle ABC: draw DE, EF, FD; the triangle DEF is one-fourth of the triangle ABC. The triangles DBE, FBE are equal, each being one-fourth of the triangle ABC: DF is therefore parallel to BE, and DBEF is a parallelogram of which DE is a diagonal.

99. This may be proved by applying Euc. I. 38.

100. Apply Euc. I. 37, 38.

101. On any side BC of the given triangle ABC, take BD equal to the *given base*; join AD, through C draw CE parallel to AD, meeting BA produced if necessary in E, join ED; then BDE is the triangle required. By a process somewhat similar the triangle may be formed when the *altitude* is given.

102. Apply the preceding problem (101) to make a triangle equal to one of the given triangles and of the same altitude as the other given triangle. Then the sum or difference can be readily found.

103. First construct a triangle on the given base equal to the given triangle; next form an isosceles triangle on the same base equal to this triangle.

104. Make an isosceles triangle equal to the given triangle, and then this isosceles triangle into an equal equilateral triangle.

105. Make a triangle equal to the given parallelogram upon the given line, and then a triangle equal to this triangle, having an angle equal to the given angle.

106. If the figure ABCD be one of four sides; join the opposite angles A, C of the figure, through D draw DE parallel to AC meeting BC produced in E, join AE:—the triangle ABE is equal to the four-sided figure ABCD.

If the figure ABCDE be one of five sides, produce the base both ways, and the figure may be transformed into a triangle, by two constructions similar to that employed for a figure of four sides. If the figure consists of six, seven, or any number of sides, the same process must be repeated.

107. Draw two lines from the bisection of the base parallel to the two sides of the triangle.

108. This may be shewn ex absurdo.

109. On the same base AB, and on the same side of it, let two triangles *ABC*, ABD be constructed, having the side BD equal to BC, the angle *ABC a right angle, but* the angle ABD not a right angle; then the triangle *ABC is greater than* ABD, whether the angle ABD be acute or obtuse.

110. Let ABC be a triangle whose vertical angle is A, and whose base BC is bisected in D; let any line EDG be drawn through D, meeting AC the greater side in G and AB produced in E, and forming a triangle AEG having the same vertical angle A. Draw BH parallel to AC, and the triangles BDH, GDC are equal. Euc. I. 26.

111. Let two triangles be constructed on the same base with equal perimeters, of which one is isosceles. Through the vertex of that which is not isosceles draw a line parallel to the base, and intersecting the perpendicular drawn from the vertex of the isosceles triangle upon the common base. Join this point of intersection and the extremities of the base.

112. (1) DF bisects the triangle ABC (fig. Prop. 6, p. 73.) On each side of the point F in the line BC, take FG, FH, each equal to one-third of BF, the lines DG, DH shall trisect the triangle. Or,

Let ABC be any triangle, D the given point in BC. Trisect BC in E, F. Join AD, and draw EG, FH parallel to AD. Join DG, DH; these lines trisect the triangle. Draw AE, AF and the proof is manifest.

(2) Let ABC be any triangle; trisect the base BC in D, E, and join AD, AE. From D, E, draw DP, EP parallel to AB, AC and meeting in P. Join AP, BP, CP; these three lines trisect the triangle.

(3) Let P be the given point within the triangle ABC. Trisect the base BC in D, E. From the vertex A draw AD, AE, AP. Join PD, draw AG parallel to PD and join PG. Then BGPA is one-third of the triangle. The problem may be solved by trisecting either of the other two sides and making a similar construction.

113. The base may be divided into nine equal parts, and lines may be drawn from the vertex to the points of division. Or, the sides of the triangle may be trisected, and the points of trisection joined.

114. It is proved, Euc. I. 34, that each of the diagonals of a parallelogram bisects the figure, and it may be shewn that they also bisect each other. It is hence manifest that any straight line, whatever may be its position, which bisects a parallelogram, *must* pass through the intersection of the diagonals.

115. See the remark on the preceding problem 114.

116. Trisect the side AB in E, F, and draw EG, FH parallel to AD or BC, meeting DC in G and H. If the given point P be in EF, the two lines drawn from P through the bisections of EG and FH will trisect the parallelogram. If P be in FB, a line from P through the bisection of FH will cut off one-third of the parallelogram, and the remaining trapezium is to be bisected by a line from P, one of its angles. If P coincide with E or F, the solution is obvious.

117. Construct a right-angled parallelogram by Euc. I. 44, equal to the given quadrilateral figure, and from one of the angles, draw a line to meet the opposite side and equal to the base of the rectangle, and a line from the adjacent angle parallel to this line will complete the rhombus.

118. Bisect BC in D, and through the vertex A, draw AE parallel to BC, with center D and radius equal to half the sum of AB, AC, describe a circle cutting AE in E.

119. Produce one side of the square till it becomes equal to the diagonal, the line drawn from the extremity of this produced side and parallel to the adjacent side of the square, and meeting the diagonal produced, determines the point required.

120. Let fall upon the diagonal perpendiculars from the opposite angles of the parallelogram. These perpendiculars are equal, and each pair of titudes. If the point be not on the diagonal, draw through the given point, a line parallel to a side of the parallelogram.

121. One case is included in Theo. 120. The other case, when the point is in the diagonal produced, is obvious from the same principle.

122. The triangles DCF, ABF may be proved to be equal to half of the parallelogram by Euc. I. 41.

123. Apply Euc. I. 41, 38.

124. If a line be drawn parallel to AD through the point of intersection of the diagonal, and the line drawn through O parallel to AB; then by Euc. I. 43, 41, the truth of the theorem is manifest.

125. It may be remarked that parallelograms, are divided into pairs of equal triangles by the diagonals, and therefore by taking the triangle ABD equal to the triangle ABC, the property may be easily shewn.

126. The triangle ABD is one half of the parallelogram ABCD, Euc. I. 34. And the triangle DKC is one half of the parallelogram CDHG, Euc. I. 41, also for the same reason the triangle AKB is one half of the parallelogram AHGB: therefore the two triangles DKC, AKB are together one half of the whole parallelogram ABCD. Hence the two triangles DKC, AKB are equal to the triangle ABD: take from these equals the equal parts which are common, therefore the triangle CKF is equal to the triangles AHK, KBD: wherefore also taking AHK from these equals, then the difference of the triangles CKF, AHK is equal to the triangle KBC: and the doubles of these are equal, or the difference of the parallelograms CFKG, AHKE is equal to twice the triangle KBD.

127. First prove that the perimeter of a square is less than the perimeter of an equal rectangle: next, that the perimeter of the rectangle is less than the perimeter of any other equal parallelogram.

128. This may be proved by shewing that the area of the isosceles triangle is greater than the area of any other triangle which has the same vertical angle, and the sum of the sides containing that angle is equal to the sum of the equal sides of the isosceles triangle.

129. Let ABC be an isosceles triangle (fig. Euc. I. 42), AE perpendicular to the base BC, and AECG the equivalent rectangle. Then AC is greater than AE, &c.

130. Let the diagonal AC bisect the quadrilateral figure ABCD. Bisect AC in E, join BE, ED, and prove BE, ED in the same straight line and equal to one another.

131. Apply Euc. I. 15.

132. Apply Euc. I. 20.

133. This may be shewn by Euc. I. 20.

134. Let AB be the longest and CD the shortest side of the rectangular figure. Produce AD, BC to meet in E. Then by Euc. I. 32.

135. Let ABCD be the quadrilateral figure, and E, F, two points in the opposite sides AB, CD, join EF and bisect it in G; and through G draw a straight line HGK terminated by the sides AD, BC; and bisected in the point G. Then EF, HK are the diagonals of the required parallelogram.

136. After constructing the figure, the proof offers no difficulty.

137. If any line be assumed as a diagonal, if the four given lines taken two and two be always greater than this diagonal, a four-sided figure may be constructed having the assumed line as one of its diagonals: and it may be shewn that when the quadrilateral is possible, the sum of every three given sides is greater than the fourth.

138. Draw the two diagonals, then four triangles are formed, two on one side of each diagonal. Then two of the lines drawn through the points of bisection of two sides may be proved parallel to one diagonal, and two

parallel to the other diagonal, in the same way as Theo. 97, supra. The other property is manifest from the relation of the areas of the triangles made by the lines drawn through the bisections of the sides.

139. It is sufficient to suggest, that triangles on equal bases, and of equal altitudes, are equal.

140. Let the side AB be parallel to CD, and let AB be bisected in E and CD in F, and let EF be drawn. Join AF, BF, then Euc. I. 38.

141. Let BCED be a trapezium of which DC, BE are the diagonals intersecting each other in G. If the triangle DBG be equal to the triangle EGC, the side DE may be proved parallel to the side BC, by Euc. I. 39.

142. Let ABCD be the quadrilateral figure having the sides AB, CD, parallel to one another, and AD, BC equal. Through B draw BE parallel to AD, then ABED is a parallelogram.

143. Let ABCD be the quadrilateral having the side AB parallel to CD. Let E, F be the points of bisection of the diagonals BD, AC, and join EF and produce it to meet the sides AD, BC in G and H. Through H draw LHK parallel to DA meeting DC in L and AB produced in K. Then BK is half the difference of DC and AB.

144. (1) Reduce the trapezium ABCD to a triangle BAE by Prob. 106, supra, and bisect the triangle BAE by a line AF from the vertex. If F fall without BC, through F draw FG parallel to AC or DE, and join AG.

Or thus. Draw the diagonals AC, BD: bisect BD in E, and join AE, EC. Draw FEG parallel to AC the other diagonal, meeting AD in F, and DC in G. AG being joined, bisects the trapezium.

(2) Let E be the given point in the side AD. Join EB. Bisect the quadrilateral EBCD by EF. Make the triangle EFG equal to the triangle EAB, on the same side of EF as the triangle AB. Bisect the triangle EFG by EH. EH bisects the figure.

145. If a straight line be drawn from the given point through the intersection of the diagonals and meeting the opposite side of the square; the problem is then reduced to the bisection of a trapezium by a line drawn from one of its angles.

146. If the four sides of the figure be of different lengths, the truth of the theorem may be shewn. If, however, two adjacent sides of the figure be equal to one another, as also the other two, the lines drawn from the angles to the bisection of the longer diagonal, will be found to divide the trapezium into four triangles which are equal in area to one another. Euc. I. 38.

147. Apply Euc. I. 47, observing that the shortest side is one half of the longest.

148. Find by Euc. I. 47, a line the square on which shall be seven times the square on the given line. Then the triangle which has these two lines containing the right angle shall be the triangle required.

149. Apply Euc. I. 47.

150. Let the base BC be bisected in D, and DE be drawn perpendicular to the hypotenuse AC. Join AD: then Euc. I. 47.

151. Construct the figure, and the truth is obvious from Euc. I. 47.

152. See Theo. 32, p. 76, and apply Euc. I. 47.

153. Draw the lines required and apply Euc. I. 47.

154. Apply Euc. I. 47.

155. Apply Euc. I. 47.

156. Apply Euc. I. 47, observing that the square on any line is four times the square on half the line.

157. Apply Euc. I. 47, to express the squares of the three sides in terms of the squares on the perpendiculars and on the segments of AB.

158. By Euc. I. 47. bearing in mind that the square described on any line is four times the square described upon half the line.

159. The former part is at once manifest by Euc. I. 47. Let the diagonals of the square be drawn, and the given point be supposed to coincide with the intersection of the diagonals, the minimum is obvious. Find its value in terms of the side.

160. (*a*) This is obvious from Euc. I. 13.

(*b*) Apply Euc. I. 32, 29.

(*c*) Apply Euc. I. 5, 29.

(*d*) Let AL meet the base BC in P, and let the perpendiculars from F, K meet BC produced in M and N respectively; then the triangles APB, FMB may be proved to be equal in all respects, as also APC, CKN.

(*e*) Let fall DQ perpendicular on FB produced. Then the triangle DQB may be proved equal to each of the triangles ABC, DBF; whence the triangle DBF is equal to the triangle ABC.

Perhaps however the better method is to prove at once that the triangles ABC, FBD are equal, by shewing that they have two sides equal in each triangle, and the included angles, one the supplement of the other.

(*f*) If DQ be drawn perpendicular on FB produced, FQ may be proved to be bisected in the point B, and DQ equal to AC. Then the square on FD is found by the right-angled triangle FQD. Similarly, the square on KE is found, and the sum of the squares on FD, EK, GH will be found to be six times the square on the hypotenuse.

(*g*) Through A draw PAQ parallel to BC and meeting DB, EC produced in P, Q. Then by the right angled triangles.

161. Let any parallelograms be described on any two sides AB, AC of a triangle ABC, and the sides parallel to AB, AC be produced to meet in a point P. Join PA. Then on either side of the base BC, let a parallelogram be described having two sides equal and parallel to AP. Produce AP and it will divide the parallelogram on BC into two parts respectively equal to the parallelograms on the sides. Euc. I. 35, 36.

162. Let the equilateral triangles ABD, BCE, CAF be described on AB, BC, CA, the sides of the triangle ABC having the right angle at A.

Join DC, AK: then the triangles DBC, ABE are equal. Next draw DG perpendicular to AB and join CG: then the triangles BDG, DAG, DGC are equal to one another. Also draw AH, EK perpendicular to BC; the triangles EKH, EKA are equal. Whence may be shewn that the triangle ABD is equal to the triangle BHE, and in a similar way may be shewn that CAF is equal to CHE.

The restriction is unnecessary: it only brings AD, AE into the same line.

GEOMETRICAL EXERCISES ON BOOK II.

HINTS, &c.

6. See the figure Euc. II. 5.

7. This Problem is equivalent to the following: construct an isosceles right-angled triangle, having given one of the sides which contains the right angle.

8. *In the question* for E read D. Construct the square on AB, and the property is obvious.

9. The sum of the squares on the two parts of any line is least when the two parts are equal.

10. A line may be found the square on which is double the square on the given line. The problem is then reduced to:—having given the hypotenuse and the sum of the sides of a right-angled triangle, construct the triangle.

11. This follows from Euc. II. 5, Cor.

12. This problem is, in other words, Given the sum of two lines and the sum of their squares, to find the lines. Let AB be the given straight line, at B draw BC at right angles to AB, bisect the angle ABC by BD. On AB take AE equal to the side of the given square, and with center A and radius AE describe a circle cutting BD in D, from D draw DF perpendicular to AB, the line AB is divided in F as was required.

13. Let AB be the given line. Produce AB to C making BC equal to three times the square on AB. From BA cut off BD equal to BC; then D is the point of section such that the squares on AB and BD are double of the square on AD.

14. In the fig. Euc. II. 7. Join BF, and draw FL perpendicular on GD. Half the rectangle DB, BG, may be proved equal to the rectangle AB, BC.

Or, join KA, CD, KD, CK. Then CK is perpendicular to BD. And the triangles CBD, KBD are each equal to the triangle ABK. Hence, twice the triangle ABK is equal to the figure CBKD; but twice the triangle ABK is equal to the rectangle AB, BC; and the figure CBKD is equal to half the rectangle DB and CK, the diagonals of the squares on AB, BC. Wherefore, &c.

15. The difference between the two unequal parts may be shewn to be equal to twice the line between the points of section.

16. This proposition is only another form of stating Euc. II. 7.

17. In the figure, Theo. 7, p. 69, draw PQ, PR, PS perpendiculars on AB, AD, AC respectively: then since the triangle PAC is equal to the two triangles PAB, PAD, it follows that the rectangle contained by PS, AC, is equal to the *sum* of the rectangles PQ, AB, and PR, AD. When is the rectangle PS, AC equal to the *difference* of the other two rectangles?

18. Through E draw EG parallel to AB, and through F, draw FHK parallel to BC and cutting EG in H. Then the area of the rectangle is made up of the areas of four triangles; whence it may be readily shewn that *twice the area* of the triangle AFE, and the figure AGHK is equal to the area ABCD.

19. Apply Euc. II. 11.

20. The vertical angles at L may be proved to be equal, and each of them a right angle.

21. Apply Euc. II. 4, 11. I. 47.

22. Produce FG, DB to meet in L, and draw the other diagonal LHC, which passes through H, because the complements AG, BK are equal. Then LH may be shewn to be equal to Ff, and to Dd.

23. The common intersection of the three lines divides each into two parts, one of which is double of the other, and this point is the vertex of three triangles which have lines drawn from it to the bisection of the bases. Apply Euc. II. 12, 13.

24. Apply Theorem 3, p. 104, and Euc. I. 47.

25. This will be found to be that particular case of Euc. II. 12, in which the distance of the obtuse angle from the foot of the perpendicular,

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